MATH122 200610 Problem Set 8 Solutions DRAFT

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1. This question essentially amounts to determining whether $\mathbf{u}$ is in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, which is the same thing as asking whether the system $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{u}$ has a solution. To answer that question, we perform row reduction on the augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 4 & 5 & -4 \\ -2 & -7 & -8 & 10 \\ 4 & 9 & 6 & -7 \\ 3 & 7 & 5 & -5 \end{array}\right].$$

Adding 2 times row 1 to row 2, -4 times row 1 to row 3, and -3 times row 1 to row 4 gives

$$\left[\begin{array}{rrr|r} 1 & 4 & 5 & -4 \\ 0 & 1 & 2 & 2 \\ 0 & -7 & -14 & 9 \\ 0 & -5 & -10 & 7 \end{array}\right].$$

Adding 7 times row 2 to row 3 and 5 times row 2 to row 4 gives

$$\left[\begin{array}{rrr|r} 1 & 4 & 5 & -4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 23 \\ 0 & 0 & 0 & 17 \end{array}\right].$$

Multiplying row 3 by 1/23 gives

$$\left[\begin{array}{rrr|r} 1 & 4 & 5 & -4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 17 \end{array}\right].$$

Adding -17 times row 3 to row 4 gives

$$\left[\begin{array}{rrr|r} 1 & 4 & 5 & -4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The matrix is in row echelon form. The third row is of the form $\left[\begin{array}{ccc|c} 0 & 0 & 0 & \square \end{array}\right]$, so the system is inconsistent, so $\mathbf{u}$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, so $\mathbf{u}$ is not in the subspace generated by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

2. (a) We find the null space by solving the system $A\mathbf{x} = \mathbf{0}$. The augmented matrix for the system is

$$\left[\begin{array}{rrrr|r} -3 & 9 & -2 & -7 & 0 \\ 2 & -6 & 4 & 8 & 0 \\ 3 & -9 & -2 & -2 & 0 \end{array}\right].$$

Multiplying row 2 by 1/2 gives

$$\left[\begin{array}{rrrr|r} -3 & 9 & -2 & -7 & 0 \\ 1 & -3 & 2 & 4 & 0 \\ 3 & -9 & -2 & -2 & 0 \end{array}\right].$$

Swapping row 1 and row 2 gives

$$\left[\begin{array}{rrrr|r} 1 & -3 & 2 & 4 & 0 \\ -3 & 9 & -2 & -7 & 0 \\ 3 & -9 & -2 & -2 & 0 \end{array}\right].$$

Adding 3 times row 1 to row 2 and -3 times row 1 to row 3 gives

$$\left[\begin{array}{rrrr|r} 1 & -3 & 2 & 4 & 0 \\ 0 & 0 & 4 & 5 & 0 \\ 0 & 0 & -8 & -14 & 0 \end{array}\right].$$

Adding 2 times row 2 to row 3 gives

$$\left[\begin{array}{rrrr|r} 1 & -3 & 2 & 4 & 0 \\ 0 & 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{array}\right].$$

The system is now in row echelon form, from which we can see that there is one free variable. Continuing to put the system into reduced row echelon form, we multiply row 3 by $-1/4$ to get

$$\left[\begin{array}{rrrr|r} 1 & -3 & 2 & 4 & 0 \\ 0 & 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Adding -5 times row 3 to row 2 and -4 times row 3 to row 1 gives

$$\left[\begin{array}{cccc|c} 1 & -3 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Multiplying row 2 by 1/4 gives

$$\left[\begin{array}{cccc|c} 1 & -3 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Adding -2 times row 2 to row 1 gives

$$\left[\begin{array}{cccc|c} 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

The system is in reduced row echelon form, and the solution to the system is

$$\begin{aligned} x_1 &= 3s \\ x_2 &= s \\ x_3 &= 0 \\ x_4 &= 0, \end{aligned}$$

or, in parametric form,

$$\mathbf{x} = s \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore a basis for the null space of A, $\operatorname{Nul} A$, is

$$\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

To find a basis for the column space of A, $\operatorname{Col} A$, we look at the pivot columns in the original matrix A. The pivot columns in the reduced row echelon form of A are columns 1, 3, and 4; therefore the pivot columns in A are the same, columns 1, 3, and 4. It is possible to show that the pivot columns of A are linearly independent (you should be able to construct a proof if you are an A student), and the non-pivot columns of A can be expressed as a linear combination of the pivot columns (in fact, the column 2 is -3 times column 1, as you can see from the reduced row echelon form of A). So a basis for the column space of A is given by the pivot columns of A, namely

$$\begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -7 \\ 8 \\ -2 \end{bmatrix}.$$

Note that the column space is a subspace of $\mathbb{R}^3$ and the null space is a subspace of $\mathbb{R}^4$.

(b) This problem is exactly the same deal as the previous. We solve the system

$$\left[\begin{array}{ccccc|c} 3 & -1 & 7 & 3 & 9 & 0 \\ -2 & 2 & -2 & 7 & 5 & 0 \\ -5 & 9 & 3 & 3 & 4 & 0 \\ -2 & 6 & 6 & 3 & 7 & 0 \end{array}\right]$$

by row reduction. Adding row 2 to row 1 gives

$$\left[\begin{array}{ccccc|c} 1 & 1 & 5 & 10 & 14 & 0 \\ -2 & 2 & -2 & 7 & 5 & 0 \\ -5 & 9 & 3 & 3 & 4 & 0 \\ -2 & 6 & 6 & 3 & 7 & 0 \end{array}\right].$$

Adding 2 times row 1 to row 2, 5 times row 1 to row 3, and 2 times row 1 to row 4 gives

$$\left[\begin{array}{ccccc|c} 1 & 1 & 5 & 10 & 14 & 0 \\ 0 & 4 & 8 & 27 & 33 & 0 \\ 0 & 14 & 28 & 53 & 74 & 0 \\ 0 & 8 & 16 & 23 & 35 & 0 \end{array}\right].$$

Adding -3 times row 2 to row 3 and -2 times row 2 to row 4 gives

$$\left[\begin{array}{ccccc|c} 1 & 1 & 5 & 10 & 14 & 0 \\ 0 & 4 & 8 & 27 & 33 & 0 \\ 0 & 2 & 4 & -28 & -25 & 0 \\ 0 & 0 & 0 & -31 & -31 & 0 \end{array}\right].$$

Swapping rows 2 and 3 gives

$$\left[\begin{array}{ccccc|c} 1 & 1 & 5 & 10 & 14 & 0 \\ 0 & 2 & 4 & -28 & -25 & 0 \\ 0 & 4 & 8 & 27 & 33 & 0 \\ 0 & 0 & 0 & -31 & -31 & 0 \end{array}\right].$$

Adding -2 times row 2 to row 3 gives

$$\left[\begin{array}{ccccc|c} 1 & 1 & 5 & 10 & 14 & 0 \\ 0 & 2 & 4 & -28 & -25 & 0 \\ 0 & 0 & 0 & 83 & 83 & 0 \\ 0 & 0 & 0 & -31 & -31 & 0 \end{array}\right].$$

Multiplying row 3 by 1/83 and row 4 by 1/31 gives

$$\left[\begin{array}{ccccc|c} 1 & 1 & 5 & 10 & 14 & 0 \\ 0 & 2 & 4 & -28 & -25 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \end{array}\right].$$

Adding 1 times row 3 to row 4 gives

$$\left[\begin{array}{ccccc|c} 1 & 1 & 5 & 10 & 14 & 0 \\ 0 & 2 & 4 & -28 & -25 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The system is now in row echelon form, from which we can see that there are two free variables in the solution set. Continuing to find the reduced row echelon form, add 28 times row 3 to row 2 and -10 times row 3 to row 1 to give

$$\left[\begin{array}{ccccc|c} 1 & 1 & 5 & 0 & 4 & 0 \\ 0 & 2 & 4 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Multiply row 1 by 2 to give

$$\left[\begin{array}{ccccc|c} 2 & 2 & 10 & 0 & 8 & 0 \\ 0 & 2 & 4 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Add -1 times row 2 to row 2 to give

$$\left[\begin{array}{ccccc|c} 2 & 0 & 6 & 0 & 5 & 0 \\ 0 & 2 & 4 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Multiply row 1 and row 2 by 1/2 to give

$$R = \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 5/2 & 0 \\ 0 & 1 & 2 & 0 & 3/2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

In the above system, variables x_3 and x_5 are free; call them s and t respectively. Then the solution set of the above system is

$$\begin{aligned} x_1 &= -3s - \frac{5}{2}t \\ x_2 &= -2s - \frac{3}{2}t \\ x_3 &= s \\ x_4 &= -t \\ x_5 &= t, \end{aligned}$$

or in parametric form,

$$\mathbf{x} = s\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5/2 \\ -3/2 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

A basis for the null space is given by the two vectors that appear above, namely

$$\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5/2 \\ -3/2 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

To find the column space, we identify the pivot columns in A. The pivot columns of A occupy the same position as the pivot columns in the row reduced version of A, i.e., the coefficient matrix of R above. In the coefficient matrix of R, it is clear that the pivot columns are columns 1, 2, and 4, so the pivot columns of A are in positions 1, 2, and 4, and are

$$\begin{bmatrix} 3 \\ -2 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 3 \\ 3 \end{bmatrix}.$$

The above three columns form a basis for the column space of A. (You should double check that those vectors are linearly independent and that the other two columns of A can be expressed as linear combinations of the three vectors above. You should be able to read linear relationships among the columns directly from the null space calculations above.)

3. (a) We need to express $\mathbf{x}$ as a linear combination of $\mathbf{b}_1$ and $\mathbf{b}_2$. In other words, we need to solve the system

$$\left[\begin{array}{cc|c} 1 & -3 & -7 \\ -3 & 5 & 5 \end{array}\right].$$

To do so, we use row reduction, as usual. Adding 3 times row 1 to row 2 give

$$\left[\begin{array}{cc|c} 1 & -3 & -7 \\ 0 & -4 & -16 \end{array}\right].$$

Multiplying row 2 by $-1/4$ gives

$$\left[\begin{array}{cc|c} 1 & -3 & -7 \\ 0 & 1 & 4 \end{array}\right].$$

Adding 3 times row 2 to row 1 gives

$$\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 4 \end{array}\right].$$

This tells us that $5\mathbf{b}_1+4\mathbf{b}_2 = \mathbf{x}$. (Check!) So we can say that the B-coordinate vector of $\mathbf{x}$ is

$$[\mathbf{x}]_B = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

As a bonus, we see that there are two pivot columns in the row echelon form of the coefficient matrix $\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$ which gives a double check that the given set B really is a basis; furthermore, the system is consistent so $\mathbf{x}$ really is in the subspace spanned by B.

(b) The idea is the same in this case. We must solve the system

$$\left[\begin{array}{cc|c} -3 & 7 & 11 \\ 1 & 5 & 0 \\ -4 & -6 & 7 \end{array}\right].$$

Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 1 & 5 & 0 \\ -3 & 7 & 11 \\ -4 & -6 & 7 \end{array}\right].$$

Adding 3 times row 1 to row 2 and 4 times row 1 to row 3 gives

$$\left[\begin{array}{cc|c} 1 & 5 & 0 \\ 0 & 22 & 11 \\ 0 & 14 & 7 \end{array}\right].$$

Multiplying row 2 by $1/22$ gives

$$\left[\begin{array}{cc|c} 1 & 5 & 0 \\ 0 & 1 & 1/2 \\ 0 & 14 & 7 \end{array}\right].$$

Adding -5 times row 2 to row 1 and -14 times row 2 to row 3 gives

$$\left[\begin{array}{cc|c} 1 & 0 & -5/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{array}\right].$$

We can conclude that

$$-\frac{5}{2}\mathbf{b}_1 + \frac{1}{2}\mathbf{b}_2 = \mathbf{x}.$$

(Check!) Therefore the B-coordinate vector of $\mathbf{x}$ is

$$[\mathbf{x}]_B = \begin{bmatrix} 11 \\ 0 \\ 7 \end{bmatrix}_B = \begin{bmatrix} -5/2 \\ 1/2 \end{bmatrix}.$$

Note that $\mathbf{x}$ is a vector in $\mathbb{R}^3$ but its B-coordinate vector is in $\mathbb{R}^2$. Also note that the row reduction process allows us to double check that B is linearly independent and that $\mathbf{x}$ is in the subspace spanned by B.

4. A basis for the subspace spanned by the given vectors is exactly the same thing as a basis for the column space of the matrix formed by juxtaposing the vectors, namely

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ -1 & -3 & 2 & 4 & -8 \\ -2 & -1 & -6 & -7 & 9 \\ 5 & 6 & 8 & 7 & -5 \end{bmatrix}.$$

To find the pivot columns of the matrix we perform row reduction. Adding 1 times row 1 to row 2, 2 times row 1 to row 3, and -5 times row 1 to row 4 we obtain

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ 0 & -1 & 2 & 3 & -5 \\ 0 & 3 & -6 & -9 & 15 \\ 0 & -4 & 8 & 12 & -20 \end{bmatrix}.$$

Multiplying row 3 by $1/3$ and row 4 by $1/4$ we obtain

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ 0 & -1 & 2 & 3 & -5 \\ 0 & 1 & -2 & -3 & 5 \\ 0 & -1 & 2 & 3 & -5 \end{bmatrix}.$$

Adding 1 times row 2 to row 3 and -1 times row 2 to row 4 we obtain

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ 0 & -1 & 2 & 3 & -5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix is now in row echelon form. We see that columns 1 and 2 are pivot columns in the above, so columns 1 and 2 must have been pivot columns in the original matrix. So a basis for the column space of the original matrix is

$$\begin{bmatrix} 1 \\ -1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \\ 6 \end{bmatrix}.$$

You should check that the above vectors are linearly independent and that the other given vectors are linear combinations of those two. The linear relationships may be found easily from a basis for the null space. (How?) The dimension of the column space is just the number of vectors in a basis; in this case, the dimension is 2.

5. (a) The rank theorem tells us that the rank of the matrix plus the nullity of the matrix is equal to the number of columns of the matrix. The nullity is 3, the number of columns is 5, so the rank must be $5 - 3 = 2$.

   (b) Let the five vectors be $\mathbf{v}_1, \ldots, \mathbf{v}_5$. Form the matrix

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \end{bmatrix}.$$

The column space of A has dimension 4 which means that there there are 4 pivot columns in A. (Equivalently, we could say that the column space has four of the $\mathbf{v}_i$s as basis vectors; or we could say that the rank of A is 4.) Since there are 4 pivot columns, there must be one non-pivot column, so the system $A\mathbf{x} = \mathbf{0}$ has one free variable. (Equivalently we could say that the dimension of the null space of A is one; or we could say that a basis of the null space contains exactly one vector; or we could say that the nullity of A is one.) Setting that free variable to some non-zero value, say $s = 1$, gives a non-trivial solution $\mathbf{x}$ to the equation $A\mathbf{x} = \mathbf{0}$. In other words, we have a non-trivial linear combination $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_5\mathbf{v}_5 = \mathbf{0}$, so the $\mathbf{v}_i$s are linearly dependent.

Try cooking up an example of five vectors $\mathbf{v}_i$ in $\mathbb{R}^5$ which span a subspace of dimension 4, and checking that the five vectors are linearly independent.

   (c) We need to construct a 4×3 matrix with one pivot column. The simplest example is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Another example is

$$\begin{bmatrix} 1 & 3 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

because there is still only one pivot element and therefore only one pivot column in the matrix. Another example is

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 6 & -10 \\ 1 & 3 & -5 \\ -3 & -9 & 15 \end{bmatrix}$$

because row operations applied to a rank 1 matrix give another rank 1 matrix. Many other examples could be constructed in a similar manner. Try constructing a few examples starting with the rank 1 reduced row echelon form matrix

$$\begin{bmatrix} 0 & 1 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

What are all possible forms of rank 1 reduced row echelon matrices?