# MATH122 200610 Problem Set 10 Solutions DRAFT

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1. (a) The determinant of the coefficient matrix is
$$\Delta = \begin{vmatrix} 4 & 1 \\ 5 & 2 \end{vmatrix} = 8 - 5 = 3.$$
Then by Cramer's rule
$$x_1 = \frac{1}{\Delta}\begin{vmatrix} 6 & 1 \\ 7 & 2 \end{vmatrix} = \frac{1}{3}(12-7) = \frac{5}{3}$$
$$x_2 = \frac{1}{\Delta}\begin{vmatrix} 4 & 6 \\ 5 & 7 \end{vmatrix} = \frac{1}{3}(28-30) = -\frac{2}{3}.$$
You should check the answer by substitution into the original equations.

   (b) The determinant of the coefficient matrix is
$$\Delta = \begin{vmatrix} -5 & 3 \\ 3 & -1 \end{vmatrix} = 5 - 9 = -4.$$
Then by Cramer's rule
$$x_1 = \frac{1}{\Delta}\begin{vmatrix} 9 & 3 \\ -5 & -1 \end{vmatrix} = -\frac{1}{4}(-9+15) = -\frac{6}{4}$$
$$x_2 = \frac{1}{\Delta}\begin{vmatrix} -5 & 9 \\ 3 & -5 \end{vmatrix} = -\frac{1}{4}(25-27) = \frac{2}{4}.$$
You can reduce the fractions to lowest terms if you like, but I generally leave them alone so that the denominators are the same for all the fractions. (In this case, coincidentally, lowest terms for both of the fractions have the same denominator, but no matter.) You should check the answer by substitution into the original equations.

   (c) With systems in three variables or more, it helps to organize the calculation better than was done above. We need to calculate four determinants: the determinant of the coefficient matrix, and the determinants of the coefficient matrix with the first column replaced by the target vector, the second column replaced by the target vector, and the third column replaced by the target vector. First, the determinant of the coefficient matrix is
$$\Delta = \begin{vmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & 0 & 2 \end{vmatrix} = -1\begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} = 4$$
where the row operation of adding $-1$ times the first row to the third row was applied. Similarly,
$$\Delta_1 = \begin{vmatrix} 4 & 1 & 1 \\ 2 & 0 & 2 \\ -2 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 4 & 1 & 1 \\ 2 & 0 & 2 \\ -6 & 0 & 2 \end{vmatrix} = -1\begin{vmatrix} 2 & 2 \\ -6 & 2 \end{vmatrix} = -16$$
where $-1$ times row 1 was added to row 3,
$$\Delta_2 = \begin{vmatrix} 2 & 4 & 1 \\ -1 & 2 & 2 \\ 3 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 8 & 5 \\ -1 & 2 & 2 \\ 0 & 4 & 9 \end{vmatrix} = -(-1)\begin{vmatrix} 8 & 5 \\ 4 & 9 \end{vmatrix} = 52$$
where 2 times row 2 was added to row 1 and 3 times row 2 was added to row 3 (note that we don't always use the same row operations to simplify determinants in Cramer's rule), and
$$\Delta_3 = \begin{vmatrix} 2 & 1 & 4 \\ -1 & 0 & 2 \\ 3 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 4 \\ -1 & 0 & 2 \\ 1 & 0 & -6 \end{vmatrix} = -1\begin{vmatrix} -1 & 2 \\ 1 & -6 \end{vmatrix} = -4$$

where $-1$ times row 1 was added to row 3. Cramer's rule now tells us that

$$x_1 = \Delta_1/\Delta = -16/4 = -4$$
$$x_2 = \Delta_2/\Delta = 52/4 = 13$$
$$x_3 = \Delta_3/\Delta = -4/4 = -1.$$

You should check those answers by substituting into the original equations.

2. (a) By the Invertible Matrix Theorem, the system has a unique solution if and only if the determinant of the coefficient matrix is non-zero. (Otherwise, the system either has more than one solution, or has no solution at all.) The determinant of the coefficient matrix is

$$\Delta = \begin{vmatrix} 3s & -5 \\ 9 & 5s \end{vmatrix} = 15s^2 + 45$$

which is never zero, so the system always has a unique solution. The solution may be found by calculating the determinants

$$\Delta_1 = \begin{vmatrix} 3 & -5 \\ 2 & 5s \end{vmatrix} = 15s + 10$$

and

$$\Delta_2 = \begin{vmatrix} 3s & 3 \\ 9 & 2 \end{vmatrix} = 6s - 27.$$

Then by Cramer's rule, the unique solution to the system is given by

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{15s + 10}{15s^2 + 45}$$
$$x_2 = \frac{\Delta_2}{\Delta} = \frac{6s - 27}{15s^2 + 45}.$$

You should check that that solution is correct.

(b) The system has a unique solution if and only if the determinant

$$\Delta = \begin{vmatrix} 2s & 1 \\ 3s & 6s \end{vmatrix} = 12s^2 - 3s$$

is not zero. Factoring,

$$\Delta = 3s(4s - 1),$$

so the determinant is non-zero if and only if $s \neq 0$ and $s \neq 1/4$. In that case, the solution to the system may be found by calculating

$$\Delta_1 = \begin{vmatrix} 1 & 1 \\ 2 & 6s \end{vmatrix} = 6s - 2$$

and

$$\Delta_2 = \begin{vmatrix} 2s & 1 \\ 3s & 2 \end{vmatrix} = s.$$

Cramer's rule then tells us that the solution is

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{6s - 2}{3s(4s - 1)}$$
$$x_2 = \frac{\Delta_2}{\Delta} = \frac{s}{3s(4s - 1)}.$$

You should check that the solution is correct.

(c) The system has a unique solution if and only if

$$\Delta = \begin{vmatrix} 2 & s & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{vmatrix}$$

is non-zero. Adding $-2$ times row 3 to row 1 and $-1$ times row 3 to row 2,

$$\Delta = \begin{vmatrix} 0 & s-8 & 7 \\ 0 & -5 & 3 \\ 1 & 4 & -2 \end{vmatrix} = \begin{vmatrix} s-8 & 7 \\ -5 & 3 \end{vmatrix}$$
$$= 3s + 11$$

so the system has a unique solution if an only if $s \neq -11/3$. In that case the solution may be found by calculating

$$\Delta_1 = \begin{vmatrix} 14 & s & 3 \\ 0 & -1 & 1 \\ -28 & 4 & -2 \end{vmatrix} = \begin{vmatrix} 14 & s & 3 \\ 0 & -1 & 1 \\ 0 & 2s+4 & 4 \end{vmatrix}$$
$$= 14(-4 - (2s + 4)) = -28s - 112,$$

$$\Delta_2 = \begin{vmatrix} 2 & 14 & 3 \\ 1 & 0 & 1 \\ 1 & -28 & -2 \end{vmatrix} = \begin{vmatrix} 2 & 14 & 3 \\ 1 & 0 & 1 \\ 5 & 0 & 4 \end{vmatrix}$$
$$= -14(4 - 5) = 14,$$

and

$$\Delta_3 = \begin{vmatrix} 2 & s & 14 \\ 1 & -1 & 0 \\ 1 & 4 & -28 \end{vmatrix} = \begin{vmatrix} 2 & s & 14 \\ 1 & -1 & 0 \\ 5 & 2s+4 & 0 \end{vmatrix}$$
$$= 14(2s + 4 + 5) = 28s + 126.$$

By Cramer's rule, the solution to the system when $s \neq -11/3$ is given by

$$\begin{aligned} x_1 &= \frac{\Delta_1}{\Delta} = \frac{-28s - 112}{3s + 11} \\ x_2 &= \frac{\Delta_2}{\Delta} = \frac{14}{3s + 11} \\ x_3 &= \frac{\Delta_3}{\Delta} = \frac{28s + 126}{3s + 11}. \end{aligned}$$

You should check that the above really is a solution to the system.

3. (a) Expanding in the third row, the determinant of the given matrix is

$$\Delta = -\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 5,$$

so the matrix (call it $A$) is invertible. The matrix of minors is

$$\begin{bmatrix} -1 & 0 & 2 \\ -3 & 0 & 1 \\ 7 & -5 & -4 \end{bmatrix}.$$

The matrix of cofactors is obtained by switching the sign at every position the sum of indices of which is negative:

$$\begin{bmatrix} -1 & 0 & 2 \\ 3 & 0 & -1 \\ 7 & 5 & -4 \end{bmatrix}.$$

The adjugate matrix is obtained by transposing the matrix of cofactors:

$$\begin{bmatrix} -1 & 3 & 7 \\ 0 & 0 & 5 \\ 2 & -1 & -4 \end{bmatrix}.$$

Finally, the inverse of the given matrix is obtained by dividing the adjugate by the determinant:

$$A^{-1} = \frac{1}{5}\begin{bmatrix} -1 & 3 & 7 \\ 0 & 0 & 5 \\ 2 & -1 & -4 \end{bmatrix}.$$

(If you are feeling energetic, you can bring the scalar inside the matrix by multiplying each element of the matrix by $1/5$.) You should check that the inverse is correct by multiplying the purported inverse by the original matrix to obtain $I$. You can avoid dealing with fractions by multiplying the adjugate by the original matrix to obtain $5I$.

(b) The determinant is

$$\Delta = \begin{vmatrix} 3 & -8 & 7 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{vmatrix} = -1(-15 + 16) = -1.$$

so the given matrix $A$ is invertible. The matrix of minors is

$$\begin{bmatrix} 5 & -2 & -4 \\ 3 & -2 & -3 \\ -8 & 3 & 6 \end{bmatrix}.$$

The matrix of cofactors is

$$\begin{bmatrix} 5 & 2 & -4 \\ -3 & -2 & 3 \\ -8 & -3 & 6 \end{bmatrix}.$$

The adjugate is the transpose of the matrix of cofactors:

$$\begin{bmatrix} 5 & -3 & -8 \\ 2 & -2 & -3 \\ -4 & 3 & 6 \end{bmatrix}.$$

The inverse is obtained by dividing through by the determinant, i.e.,

$$\begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix}.$$

You should check by matrix multiplication with the original matrix.

(c) Since the matrix is upper triangular, its determinant is the product of the diagonal elements:

$$\Delta = 1(-3)(3) = -9,$$

which is non-zero, so the matrix is invertible. The matrix of minors is

$$\begin{bmatrix} -9 & 0 & 0 \\ 6 & 3 & 0 \\ 14 & 1 & -3 \end{bmatrix}.$$

The matrix of cofactors is

$$\begin{bmatrix} -9 & 0 & 0 \\ -6 & 3 & 0 \\ 14 & -1 & -3 \end{bmatrix}.$$

The adjugate is the transpose of the matrix of cofactors, namely

$$\begin{bmatrix} -9 & -6 & 14 \\ 0 & 3 & -1 \\ 0 & 0 & -3 \end{bmatrix}.$$

Finally, the inverse is

$$-\frac{1}{9}\begin{bmatrix} -9 & -6 & 14 \\ 0 & 3 & -1 \\ 0 & 0 & -3 \end{bmatrix}.$$

You should check by matrix multiplication with the original matrix. It is enough to check that the original matrix times its adjugate is $-9I$; that way you avoid dealing with fractions.

The inverse of an upper triangular matrix is always upper triangular. Can you use Cramer's rule to see why that is true?

4. (a) The linear transformation with standard matrix

$$A = \begin{bmatrix} -1 & 4 \\ 3 & -5 \end{bmatrix}$$

maps the standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ (also known as $\mathbf{i}$ and $\mathbf{j}$) to the sides of the parallelogram, so expands the area of the unit square to the area of the parallelogram. The determinant of $A$ is $-7$, so the area of the parallelgram is 7.

(b) Label the four points $P$, $Q$, $R$, and $S$ in that order. (It will help to draw a picture.) Consider the three vectors

$$PQ = \begin{bmatrix} 6 \\ 1 \end{bmatrix}, PR = \begin{bmatrix} -3 \\ 3 \end{bmatrix}, PS = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

obtained by subtracting the end point from the starting point $P$. Note that $PS = PQ + PR$, so the figure is indeed a parallelogram. Translating $P$ to the origin, the paralellogram becomes the image of the unit square under the linear transformation with standard matrix

$$\begin{bmatrix} 6 & -3 \\ 1 & 3 \end{bmatrix}.$$

Since the above matrix has determinant 21, that is therefore the area of the parallelogram. (You could do the problem by picking any of the points $P$, $Q$, $R$, or $S$ as the starting point. Picking a different starting point gives a way of checking your answer.)

(c) The volume of the parallelepiped is equal to the factor by which the transformation with standard matrix

$$\begin{bmatrix} 1 & -2 & -1 \\ 4 & -5 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

expands volumes, i.e., the volume is equal to the absolute value of the determinant $\Delta$ of the above matrix. Adding 2 times column 3 to column 2 gives

$$\Delta \begin{bmatrix} 1 & -4 & -1 \\ 4 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix} = -15.$$

Therefore the area of the parallelepiped is 15.

5. (a) We have

$$\begin{aligned} \mathbf{a}\cdot\mathbf{a} &= (\sqrt{3})^2 + 1^2 = 4 \\ \mathbf{b}\cdot\mathbf{b} &= 0^2 + 5^2 = 25 \\ \mathbf{a}\cdot\mathbf{b} &= \sqrt{3}(0) + 1(5) = 5. \end{aligned}$$

The lengths of $\mathbf{a}$ and $\mathbf{b}$ are given by

$$\begin{aligned} \|\mathbf{a}\| &= \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{4} = 2 \\ \|\mathbf{b}\| &= \sqrt{\mathbf{b}\cdot\mathbf{b}} = \sqrt{25} = 5. \end{aligned}$$

The cosine of the angle between the vectors is given by

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{5}{2\cdot 5} = \frac{1}{2}.$$

Therefore an exact expression for the angle between the two vectors is

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

Using your calculator, or a 1, 2, $\sqrt{3}$ right-angled triangle, you can obtain the numerical value

$$\theta = \frac{\pi}{3} \text{ radians } = 60 \text{ degrees.}$$

(b) Calling the first vector $\mathbf{a}$ and the second $\mathbf{b}$, we have

$$\begin{aligned} \mathbf{a}\cdot\mathbf{a} &= 6^2 + (-3)^2 + 2^2 = 49 \\ \mathbf{b}\cdot\mathbf{b} &= 2^2 + 1^2 + (-2)^2 = 9 \\ \mathbf{a}\cdot\mathbf{b} &= 6(2) + (-3)(1) + 2(-2) = 5 \end{aligned}$$

The lengths of $\mathbf{a}$ and $\mathbf{b}$ are given by

$$\begin{aligned} \|\mathbf{a}\| &= \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{49} = 7 \\ \|\mathbf{b}\| &= \sqrt{\mathbf{b}\cdot\mathbf{b}} = \sqrt{9} = 3, \end{aligned}$$

and the cosine of the angle between the vectors is

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{5}{7\cdot 3} = \frac{5}{21}.$$

Therefore an exact expression for the angle between the two vectors is

$$\theta = \cos^{-1}\left(\frac{5}{21}\right).$$

To the nearest degree, we have $\theta = 76$ degrees; to the nearest hundredth of a radian, we have $\theta = 1.33$ radians.

(c) We have

$$\mathbf{a}\cdot\mathbf{a} = 2^2 + (-1)^2 + 1^2 = 6$$
$$\mathbf{b}\cdot\mathbf{b} = 3^2 + 2^2 + (-1)^2 = 14$$
$$\mathbf{a}\cdot\mathbf{b} = 2(3) + (-1)(2) + 1(-1) = 3$$

The lengths of **a** and **b** are given by

$$\|\mathbf{a}\| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{6}$$
$$\|\mathbf{b}\| = \sqrt{\mathbf{b}\cdot\mathbf{b}} = \sqrt{14}$$

and the cosine of the angle between the vectors is

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{3}{\sqrt{6}\sqrt{14}} = \frac{3}{\sqrt{84}}.$$

Therefore an exact expression for the angle between the two vectors is

$$\theta = \cos^{-1}\left(\frac{3}{\sqrt{84}}\right).$$

To the nearest degree we have $\theta = 71$ degrees; to the nearest hundredth of a radian we have $\theta = 1.24$ radians.

6. In each case, the vectors are orthogonal if and only if the angle between them is $\pi/2$ radians or 90 degrees, i.e., if and only if the cosine of the angle between them is 0, i.e., if and only if the dot product of the vectors is 0. The vectors are parallel if and only if the angle between them is 0 radians (0 degrees) or $\pi$ radians (180 degrees), i.e., if and only if the cosine of the angle between them is $\pm 1$, i.e., if and only if

$$\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \pm 1.$$

We will test for orthogonality first, and if that fails, we will test whether the vectors are parallel.

(a) The dot product of the vectors is $-3(4) + 9(-12) + 6(-8) = -168$, so the vectors are not orthogonal. The length of the first vector is $\sqrt{126}$, and the length of the second vector is $\sqrt{224}$. We have $\cos\theta = -168/(\sqrt{126\cdot 224}) = -1$ so the vectors are parallel.

(b) The dot product of the vectors is $2 + 1 + 2 = 5$, so the vectors are not orthogonal. The length of **a** is $\sqrt{6}$; the length of **b** is $\sqrt{6}$; the cosine of the angle between the vectors is $5/(\sqrt{6}\sqrt{6}) = 5/6$ which is neither 1 nor $-1$, so the vectors are not parallel either.

(c) The dot product of the vectors is $a(-b) + b(a) + c(0) = -ab + ab = 0$ no matter what $a, b, c$ are, so the vectors are always orthogonal. (An exception is the case where $a = b = 0$; the concept of orthogonality doesn't apply when one of the vectors is $\mathbf{0}$.)

7. The direction cosines are determined by forming a unit vector from the given vector; that is, dividing the vector by its length to obtain a vector pointing in the same direction as the given vector, but having unit length. The direction angles are determined by taking the inverse cosine of the direction cosines.

(a) Calling the given vector **a**, the length of **a** is

$$\|\mathbf{a}\| = (1^2 + (-2)^2 + (-1)^2)^{1/2} = \sqrt{6}$$

so the unit vector pointing in the same direction as **a** is

$$\frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}.$$

Therefore the direction cosines are

$$\begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

and the corresponding direction angles

are

$$\begin{aligned}\alpha &= \cos^{-1}\left(\frac{1}{\sqrt{6}}\right)\\ &\approx 66 \text{ degrees } = 1.15 \text{ radians},\\ \beta &= \cos^{-1}\left(\frac{-2}{\sqrt{6}}\right)\\ &\approx 154 \text{ degrees } = 2.53 \text{ radians},\\ \gamma &= \cos^{-1}\left(\frac{-1}{\sqrt{6}}\right)\\ &\approx 114 \text{ degrees } = 1.99 \text{ radians}.\end{aligned}$$

(b) The length of the given vector is 3 so the direction cosines are

$$\cos\alpha = \frac{2}{3},\quad \cos\beta = -\frac{1}{3},\quad \cos\gamma = \frac{2}{3},$$

and the direction angles are

$$\begin{aligned}\alpha &= \cos^{-1}\left(\frac{2}{3}\right)\\ &\approx 48 \text{ degrees } = 0.84 \text{ radians},\\ \beta &= \cos^{-1}\left(\frac{-1}{3}\right)\\ &\approx 109 \text{ degrees } = 1.91 \text{ radians},\\ \gamma &= \cos^{-1}\left(\frac{2}{3}\right)\\ &\approx 48 \text{ degrees } = 0.84 \text{ radians},\end{aligned}$$

(c) At first sight, one might think that the direction cosines would depend on $c$, but that isn't the case. (No matter what $c$ is, the vector points in the same direction as $-\mathbf{i}+\mathbf{j}-\mathbf{k}$ so the direction angles (and therefore direction cosines) shouldn't depend on $c$. In detail, if the vector is called $\mathbf{a}$, we have

$$\begin{aligned}||\mathbf{a}|| &= (\mathbf{a}\cdot\mathbf{a})^{1/2}\\ &= ((-c)^2 + c^2 + (-c)^2)^{1/2} = (3c^2)^1/2\\ &= \sqrt{3}c\end{aligned}$$

so the unit vector in the same direction as $\mathbf{a}$ is

$$\mathbf{u} = \frac{1}{\sqrt{3}c}\begin{bmatrix}-c\\ c\\ -c\end{bmatrix} = \begin{bmatrix}-1/\sqrt{3}\\ 1/\sqrt{3}\\ -1\sqrt{3}\end{bmatrix}$$

so the direction cosines are

$$\cos\alpha = -\frac{1}{\sqrt{3}},\quad \cos\beta = \frac{1}{\sqrt{3}},\quad \cos\gamma = -\frac{1}{\sqrt{3}}$$

(note they do not depend on $c$) and the direction angles are

$$\begin{aligned}\alpha &= 125 \text{ degrees } = 2.19 \text{ radians},\\ \beta &= 55 \text{ degrees } = 0.96 \text{ radians},\\ \gamma &= 125 \text{ degrees } = 2.19 \text{ radians}.\end{aligned}$$

What happens if $c$ is negative?

8. The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ (also called the component of $\mathbf{b}$ in the $\mathbf{a}$ direction) is the length of the adjacent side of the right triangle with hypoteneuse $\mathbf{b}$ and adjacent side parallel to $\mathbf{a}$. By elementary trigonometry, the length of interest is the length of the hypoteneuse times the cosine of the angle between the vectors, so is

$$\text{comp}_{\mathbf{a}}\,\mathbf{b} = ||\mathbf{b}||\frac{\mathbf{a}\cdot\mathbf{b}}{||a||||b||} = \frac{\mathbf{a}\cdot\mathbf{b}}{||a||},$$

the formula given in the Stewart handout. The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is a vector in the same direction as $\mathbf{a}$ with length equal to the scalar projection; it can be found by multiplying a unit vector in the $\mathbf{a}$ direction by the scalar projection, i.e.

$$\text{proj}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||}\frac{\mathbf{a}}{||\mathbf{a}||} = \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||^2}\mathbf{a}$$

again agreeing with the formula given in the Stewart handout. In order to calculate these values, we need to find the dot product of $\mathbf{a}$ and $\mathbf{b}$, the length of $\mathbf{a}$, and (for the vector projection) a unit vector in the $\mathbf{a}$ direction.

(a) In this case we have $\mathbf{a}\cdot\mathbf{b} = -2$, $||\mathbf{a}|| = \sqrt{5}$, so

$$\text{comp}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||} = \frac{-2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}.$$

The unit vector in the $\mathbf{a}$ direction is

$$\mathbf{u} = \frac{\mathbf{a}}{||\mathbf{a}||} = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\ 2\end{bmatrix}$$

so the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\begin{aligned}\operatorname{proj}_{\mathbf{a}} \mathbf{b} &= (\operatorname{comp}_{\mathbf{a}} \mathbf{b})\mathbf{u} \\ &= \left(-\frac{2\sqrt{5}}{5}\right)\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -2/5 \\ -4/5 \end{bmatrix}.\end{aligned}$$

(b) The dot product of the vectors is $-1$, the length of $\mathbf{a}$ is 3, so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\operatorname{comp}_{\mathbf{a}} \mathbf{b} = -\frac{1}{3}.$$

The unit vector in the $\mathbf{a}$ direction is

$$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \begin{bmatrix} -1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

so the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\operatorname{proj}_{\mathbf{a}} \mathbf{b} = -\frac{1}{3}\begin{bmatrix} -1/3 \\ -2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1/9 \\ 2/9 \\ -2/9 \end{bmatrix}.$$

(c) The dot product of the vectors is $-18$, the length of $\mathbf{a}$ is $\sqrt{14}$, so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $-18/\sqrt{14} = -9\sqrt{14}/7$. The unit vector in the $\mathbf{a}$ direction is

$$\mathbf{u} = \frac{2}{\sqrt{14}}\mathbf{i} - \frac{3}{\sqrt{14}}\mathbf{j} + \frac{1}{\sqrt{14}}\mathbf{k}$$

so the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\operatorname{proj}_{\mathbf{a}} \mathbf{b} = -\frac{18}{7}\mathbf{i} + \frac{27}{7}\mathbf{j} - \frac{9}{7}\mathbf{k}.$$

9. (a) The cross product may be found using the mnemonic

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 1 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 2\mathbf{i} - 14\mathbf{j} + \mathbf{k}.$$

The dot product of the above vector with $\mathbf{a}$ is $5(2) + 1(-14) + 4(1) = 0$, so the cross product is orthogonal to $\mathbf{a}$. The dot product of the above vector with $\mathbf{b}$ is $-1(2) + 0(-14) + 2(1) = 0$, so the cross product is orthogonal to $\mathbf{b}$. (The latter technique provides a useful check that the cross product calculation is correct.)

(b) The cross product is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{k}.$$

The dot product with $\mathbf{a}$ is $1(-2) - 1(0) + 1(2) = 0$, and the dot product with $\mathbf{b}$ is $1(-2) + 1(0) + 1(2) = 0$, so $\mathbf{a} \times \mathbf{b} = -2\mathbf{i} + 2\mathbf{k}$ checks.

10. By *unit vector*, we mean a vector with unit length. We start by finding a (not necessarily unit) vector $\mathbf{v}$ perpendicular to both the given vectors, and then forming a unit vector $\mathbf{u}$ from $\mathbf{v}$. To obtain a second unit vector we form $-\mathbf{u}$. In detail, a vector perpendicular to both given vectors is

$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}.$$

You should check that $\mathbf{v}$ really is orthogonal to both $\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $2\mathbf{i}+\mathbf{k}$ by forming the appropriate dot products. Next, we find a unit vector $\mathbf{u}$ in the same direction as $\mathbf{v}$ by dividing $\mathbf{v}$ by its length:

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{6}}(\mathbf{i} + \mathbf{j} - 2\mathbf{k}).$$

A second unit vector perpendicular to both given vectors is

$$-\mathbf{u} = -\frac{1}{\sqrt{6}}(\mathbf{i} + \mathbf{j} - 2\mathbf{k}).$$

You should check that the latter vector has the necessary properties (unit length and orthogonal to both given vectors).

11. (a) Choosing $A$ as the starting point, the vectors connecting $A$ to the other given points are

$$AB = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, AC = \begin{bmatrix} 6 \\ 1 \end{bmatrix}, AD = \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

Since $AB + AD = AC$, the figure really is a parallelogram, with vertex $C$ opposite vertex $A$ (sketch a diagram). In order to use the cross product to find the area of the parallelogram, we need to lift the figure into three dimensional space. We do so by pulling the figure into the $x_1x_2$-plane in $\mathbb{R}^3$, i.e., we set the third component of the vectors to 0: two adjacent edges of the parallelogram are given by

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}.$$

According to a result in the textbook, the area of the parallelogram with sides $\mathbf{v}_1$ and $\mathbf{v}_2$ is the length of the cross product of the two vectors, i.e., $\|\mathbf{v}_1 \times \mathbf{v}_2\|$. Finding the cross product,

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ 4 & -2 & 0 \end{vmatrix} = -16\mathbf{k}.$$

The length of the cross product is 16, which is therefore the area of the parallelogram in question. (Note that we could have obtained the same answer by taking the determinant with columns $AB$ and $AD$. The advantage to the cross product method is that it works even when the sides of the parallelogram do not lie in the $x_1x_2$-plane.)

(b) According to the textbook, the volume of the parallelepiped is equal to the absolute value of the triple product $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$. Forming the cross product of the first two vectors,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = -2\mathbf{j} - 2\mathbf{k}.$$

You should check that the above vector is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. Next, we form the dot product with the third vector $\mathbf{c} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$:

$$\dot{\mathbf{c}}(\mathbf{a} \times \mathbf{b}) = -1(0) + 1(-2) + 1(-2) = -4$$

The absolute value of the triple product is $|\dot{\mathbf{c}}(\mathbf{a} \times \mathbf{b})| = 4$, which is therefore the volume of the parallelepiped. Note that we could have found the volume more quickly by forming the appropriate determinant from the three given vectors. The triple product is just an alternative way of representing the determinant and structuring the calculation.

(c) The adjacent sides of the parallelepiped are the vectors

$$PQ = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, PR = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, PS = \begin{bmatrix} 6 \\ -2 \\ 2 \end{bmatrix}.$$

According to the textbook, the volume of the parallelepiped is the absolute value of the triple product $PS \cdot (PQ \times PR)$. Taking the cross product,

$$PQ \times PR = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 3 \\ -1 & -1 & -1 \end{vmatrix} = -\mathbf{j} + \mathbf{k}.$$

Taking the dot product with $PS$,

$$PS \cdot (PQ \times PR) = 6(0) + (-2)(-1) + 2(1) = 4,$$

so the volume of the parallelepiped is 4. You can check by calculating the volume as a determinant, or by calculating any of the other cross products (e.g., $PR \cdot (PQ \times PS)$).

12. Two vectors in the plane are $PQ$ and $PR$, so any vector orthogonal to both $PQ$ and $PR$ will be orthogonal to the plane. The cross product is one such vector. As a bonus, the length of the cross product is equal the area of the parallelogram determined by the two vectors, which is twice the area of the triangle with those two vectors as sides (draw a picture to see why). So we can answer both questions at once by finding the cross product of two sides of the triangle.

    (a) In this case,

    $$PQ = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}, PR = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

    so their cross product is

    $$PQ \times PR = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}.$$

    (As a check, you should verify that the above vector is orthogonal to both $PQ$ and $PR$, and also to the third side of the triangle, $QR$.) We now have a vector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ which is orthogonal to the plane containing $P$, $Q$, and $R$. Furthermore, the area of the parallelogram defined by $PQ$ and $PR$ is the length of the cross product, i.e., $\sqrt{6}$, so the area of the triangle $PQR$ is half of that, i.e., $\sqrt{6}/2$.

    (b) In this case, $PQ = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$, $PR = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$, $PQ \times PR = -\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ is a vector orthogonal to the plane containing $PQR$, and the area of $PQR$ is $\sqrt{18}/2$.