MATH122 200610 Problem Set 11 Solutions DRAFT

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1. (a) Let the given matrix be A and the given vector be $\mathbf{v}$. Performing the matrix multiplication,

$$\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -2+2\sqrt{2}+1 \\ -1+\sqrt{2}+4 \end{bmatrix} = \begin{bmatrix} -1+2\sqrt{2} \\ 3+\sqrt{2} \end{bmatrix}.$$

It is not immediately obvious whether $A\mathbf{v}$ is a scalar multiple of $\mathbf{v}$. We could calculate the angle between $\mathbf{v}$ and $A\mathbf{v}$ (the angle must be 0 or π radians in order for the vectors to be parallel), but that probably harder than necessary for the problem at hand. Instead, note that if $A\mathbf{v}$ is a multiple of $\mathbf{v}$, the eigenvalue must be $3+\sqrt{2}$ by comparing the second entry of each vector. Multiplying the first entry of $\mathbf{v}$ by $3+\sqrt{2}$ we have

$$(3+\sqrt{2})(-1+\sqrt{2}) = -3+3\sqrt{2}-\sqrt{2}+2 = -1+2\sqrt{2}$$

which is equal to the first entry of $A\mathbf{v}$. Therefore $A\mathbf{v} = \lambda\mathbf{v}$ with $\lambda = 3+\sqrt{2}$, i.e., $\mathbf{v}$ is an eigenvector of A with eigenvalue $3+\sqrt{2}$. You could check your answer by finding the characteristic equation of A, checking that λ satisfies the characteristic equation, and then checking that $\mathbf{v}$ is in the null space of $A-\lambda I$.

(b) Let the matrix be A and the vector be $\mathbf{v}$. Performing the matrix multiplication,

$$\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}.$$

If $\mathbf{v}$ is an eigenvector of A, then comparing first entries of $\mathbf{v}$ and $A\mathbf{v}$, we see that the eigenvalue must be -2. Now comparing $A\mathbf{v}$ and $-2\mathbf{v}$, we see that the two are equal so $\mathbf{v}$ is an eigenvector of A with eigenvalue -2. You can check that $A+2I$ is not invertible and that $\mathbf{v}$ is in the null space of $A+2I$.

(c) Let the matrix be A and the vector be $\mathbf{v}$. Performing the matrix multiplication,

$$\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \\ 12 \end{bmatrix},$$

so $\mathbf{v}$ is an eigenvector of A with eigenvalue 6. You can check that $A-6I$ is not invertible and that $\mathbf{v}$ is in the null space of $A-6I$.

In all three cases the given vector was an eigenvector of the given matrix. However, if you pick a matrix and vector at random, the vector will almost surely not be an eigenvector of the matrix. Try it!

2. In each case we determine whether $A-\lambda I$ is not invertible, and if it is not, we find one vector in the null space. It is usually most efficient to use row reduction for this problem, although you could use determinants if you wanted.

(a) We perform row reduction on the system

$$\left[\begin{array}{cc|c} 9 & 3 & 0 \\ 3 & 1 & 0 \end{array}\right]$$

Multiplying row 1 by $1/9$ and then adding -3 times row 1 to row 2 gives

$$\left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Since the row reduced coefficient matrix has a row of zeros, the original coefficient matrix is not invertible, and -2 is an eigenvalue of the given matrix. To find an eigenvector, we find any solution to the above system. For example,

$$\mathbf{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

works. (I prefer integer answers where possible, so I took $s = 3$.) You should double check that the above vector is an eigenvector of the given matrix with eigenvalue -2.

(b) We perform row reduction on the system

$$\left[\begin{array}{rrr|r} -2 & 2 & 2 & 0 \\ 3 & -5 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right].$$

Adding 1 times row 2 to row 1 gives

$$\left[\begin{array}{rrr|r} 1 & -3 & 3 & 0 \\ 3 & -5 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right].$$

Adding -3 times row 1 to row 2 gives

$$\left[\begin{array}{rrr|r} 1 & -3 & 3 & 0 \\ 0 & 4 & -8 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right].$$

Multiplying row 2 by $1/4$ and then adding -1 times row 2 to row 3 gives

$$\left[\begin{array}{rrr|r} 1 & -3 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The matrix is in row echelon form, and we can see that it has only two pivot columns, so the original coefficient matrix is not invertible. It follows that 3 is an eigenvalue of the given matrix. To find a corresponding eigenvector, we put the system in reduced row echelon form. Adding 3 times row 2 to row 1 gives

$$\left[\begin{array}{rrr|r} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Choosing an appropriate value for the free variable gives us the eigenvector

$$\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

You should check that $\mathbf{v}$ really is an eigenvector of the given matrix with the appropriate eigenvalue.

(c) We perform row reduction on the system

$$\left[\begin{array}{rrr|r} 0 & 2 & 3 & 0 \\ -1 & -3 & -3 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right].$$

Adding 2 times row 2 to row 3, then multiplying row 2 by -1 and swapping rows 1 and 2 gives

$$\left[\begin{array}{rrr|r} 1 & 3 & 3 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right].$$

Adding 1 times row 3 to row 2 and swapping rows 2 and 3 gives

$$\left[\begin{array}{rrr|r} 1 & 3 & 3 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right].$$

The matrix is now in row echelon form, from which we can see that the coefficient matrix has a pivot in each column, so is invertible. Therefore 4 is not an eigenvalue of the given matrix. You could check by finding the characteristic polynomial of the matrix and verifying that 4 is not a root of the characteristic polynomial.

3. We solve this problem just as we solved the previous problem.

(a) We need to find a basis for the null space of

$$\begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix},$$

i.e., we need to find a basis for the solution space of

$$\left[\begin{array}{rr|r} 6 & -9 & 0 \\ 4 & -6 & 0 \end{array}\right],$$

Multiplying row 1 by $1/3$ and row 2 by $1/2$ gives

$$\left[\begin{array}{rr|r} 2 & -3 & 0 \\ 2 & -3 & 0 \end{array}\right].$$

Adding -1 times row 1 to row 2 and then multiplying row 1 by $1/2$ gives

$$\left[\begin{array}{rr|r} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Writing $s = 2s'$ to ensure that we are working with integers and not fractions, the solution set is

$$x_1 = 3s'$$
$$x_2 = 2s'$$

or in parametric form,

$$\mathbf{x} = s' \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

So a basis for the eigenspace associated with the eigenvalue 4 is

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

You can check that the above vector really is an eigenvector for the given matrix.

(b) We need to solve

$$\left[\begin{array}{rrr|r} 3 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 4 & -13 & 3 & 0 \end{array}\right].$$

Adding -3 times row 2 to row 1, -4 times row 2 to row 3, and then swapping rows 1 and 2 gives

$$\left[\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & -9 & 3 & 0 \end{array}\right].$$

Adding 3 times row 2 to row 3 and then multiplying row 2 by 1/3 gives

$$\left[\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Adding 1 times row 2 to row 1 gives

$$\left[\begin{array}{rrr|r} 1 & 0 & -1/3 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Letting $s = 3s'$ so we have an integer solution,

$$x_1 = s'$$
$$x_2 = s'$$
$$x_3 = 3s'$$

or in parametric form

$$\mathbf{x} = s' \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}.$$

So a basis for the eigenspace in question is

$$\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}.$$

You should check that the above vector really is an eigenvector of the given matrix with the appropriate eigenvalue.

(c) We need to solve the system

$$\left[\begin{array}{rrrr|r} -1 & 0 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Adding 1 times row 1 to row 2 and then multiplying row 1 by -1 gives

$$\left[\begin{array}{rrrr|r} 1 & 0 & -2 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Adding 1 times row 2 to row 3 and then multiplying row 1 by -1 gives

$$\left[\begin{array}{rrrr|r} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The system is now in reduced row echelon form. There are two free variables, $x_3 = s$ and $x_4 = t$, and the general solution is

$$x_1 = 2s$$
$$x_2 = 3s$$
$$x_3 = s$$
$$x_4 = t,$$

or in parametric form,

$$\mathbf{x} = s \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore a basis for the eigenspace associated with the eigenvalue 4 is

$$\begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

You should check that both of the above vectors are eigenvectors for the given matrix with eigenvalue 4, and that both of the above vectors are linearly independent.

4. (a) The characteristic polynomial of the matrix is

$$\begin{aligned}
p(\lambda) &= \det(A - \lambda I) \\
&= \det\left(\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\
&= \det\begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix} \\
&= (5-\lambda)^2 - 9 \\
&= \lambda^2 - 10\lambda + 16 \\
&= (\lambda - 2)(\lambda - 8).
\end{aligned}$$

Therefore the eigenvalues are 2 and 8. As a check you can try to find corresponding eigenvectors as in the previous two problems.

(b) The characteristic polynomial is

$$\begin{vmatrix} 3-\lambda & -4 \\ 4 & 8-\lambda \end{vmatrix} = \lambda^2 - 11\lambda + 40.$$

The characteristic polynomial can't be factored in an obvious way so we try the quadratic formula:

$$\begin{aligned}
\lambda &= \frac{11 \pm \sqrt{(-11)^2 - 4(1)(40)}}{2(1)} \\
&= \frac{11 \pm \sqrt{-39}}{2}.
\end{aligned}$$

Since the discriminant $b^2 - 4ac$ is negative, the characteristic polynomial has no real roots, so the matrix has no (real) eigenvalues.

(c) The characteristic polynomial is

$$\begin{vmatrix} 7-\lambda & -2 \\ 2 & 3-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 25.$$

Then either by factoring in integers, or by completing the square, or by the quadratic formula, the characteristic polynomial factors as

$$\lambda^2 - 10\lambda + 25 = (\lambda - 5)^2.$$

The characteristic polynomial has double root $\lambda = 5, 5$, so 5 is the only eigenvalue of the given matrix, with (algebraic) multiplicity 2. You should check the result by finding an associated eigenvector.

5. As in the previous problem, the characteristic polynomial is $\det(A - \lambda I)$, which is now a 3×3 determinant and a degree 3 (cubic) polynomial. In general it is a very hard problem to factor cubic polynomials, but where possible I will do so, because it facilitates checking; if you can find the eigenvalues explicitly, you can try to find eigenvectors, which provides a check that you are correct.

(a) Expanding in the first row, the characteristic polynomial is

$$\begin{aligned}
p(\lambda) &= \det(A - \lambda I) \\
&= \begin{vmatrix} -\lambda & 3 & 1 \\ 3 & -\lambda & 2 \\ 1 & 2 & -\lambda \end{vmatrix} \\
&= -\lambda \begin{vmatrix} -\lambda & 2 \\ 2 & -\lambda \end{vmatrix} - 3 \begin{vmatrix} 3 & 2 \\ 1 & -\lambda \end{vmatrix} + 1 \begin{vmatrix} 3 & -\lambda \\ 1 & 2 \end{vmatrix} \\
&= -\lambda(\lambda^2 - 4) - 3(-3\lambda - 2) + 1(6 + \lambda) \\
&= -\lambda^3 + 14\lambda + 12.
\end{aligned}$$

We have no obvious way of factoring the characteristic polynomial, unfortunately. The only check we have is to do the calculation over again, perhaps picking a different row or column in which to expand the determinant.

(b) Expanding in the third row, the characteristic polynomial is

$$\begin{aligned}
p(\lambda) &= \begin{vmatrix} -1-\lambda & 0 & 1 \\ -3 & 4-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} \\
&= (2-\lambda)(-1-\lambda)(4-\lambda).
\end{aligned}$$

We were lucky to get the characteristic polynomial in factored form. You can see that the eigenvalues of the matrix are $\lambda = 2, -1, 4$. You can check that those really are eigenvalues by finding a corresponding eigenvector for each eigenvalue.

(c) Expanding in the second row, the characteristic polynomial is

$$p(\lambda) = \begin{vmatrix} 5-\lambda & -2 & 3 \\ 0 & 1-\lambda & 0 \\ 6 & 7 & -2-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 5-\lambda & 3 \\ 6 & -2-\lambda \end{vmatrix}$$
$$= (1-\lambda)(\lambda^2 - 3\lambda - 28)$$
$$= (1-\lambda)(\lambda-7)(\lambda+4).$$

Because of the form of the determinant in this case, one factor of the characteristic polynomial pops out immediately, after which the remaining quadratic can be factored by standard methods. So, fortunately, in this case we can check our work by finding eigenspaces corresponding to the eigenvalues $\lambda = 1, 7, -4$.

6. (a) The characteristic polynomial and eigenvalues are easy to find for an upper triangular matrix. In this case the characteristic polynomial is

$$p(\lambda) = (-1-\lambda)(4-\lambda)(-1-\lambda)$$

which implies that the matrix has eigenvalues $\lambda = -1, -1, 4$. (The eigenvalue -1 has (algebraic) multiplicity 2.) Let's find a basis for the eigenspace associated with $\lambda = 4$. We need to find a basis for the solution set of the system

$$\left[\begin{array}{ccc|c} -5 & -2 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -5 & 0 \end{array}\right].$$

Multiplying row 2 by 1/3 and then adding appropriate multiples of row 2 to the other rows we have

$$\left[\begin{array}{ccc|c} -5 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Multiplying row 1 by $-1/5$ gives

$$\left[\begin{array}{ccc|c} 1 & 2/5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So the general solution is

$$x_1 = -\frac{2}{5}s$$
$$x_2 = s$$
$$x_3 = 0$$

Letting $s = 5s'$ so we don't have to deal with fractions, the solution set can be expressed in parametric form as

$$\mathbf{x} = s'\begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix}.$$

Therefore a basis for the eigenspace corresponding to the eigenvalue 4 is

$$\begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix}.$$

You should check that the above vector really is an eigenvector for the matrix with eigenvalue 4.

It remains to find the eigenspace corresponding to eigenvalue -1. We must solve the system

$$\left[\begin{array}{ccc|c} 0 & -2 & 2 & 0 \\ 0 & 5 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Adding 2 times row 1 to row 2 gives

$$\left[\begin{array}{ccc|c} 0 & -2 & 2 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Adding 2 times row 2 to row 1 and swapping rows 1 and 2 gives

$$\left[\begin{array}{ccc|c} 0 & 1 & 7 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Multiplying row 2 by 1/16 and adding the appropriate multiple of row 2 to row 1 gives

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The system is in reduced row echelon form. The variable x_1 is free, and $x_2 = x_3 = 0$. In parametric form the solution set is

$$\mathbf{x} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

so a basis for the eigenspace associated with the eigenvalue -1 is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

You should check that the above really is an eigenvector for the matrix with eigenvalue -1. (Note that the eigenspace associated with the eigenvalue -1 has dimension 1; another way of expressing that fact is by saying that the geometric multiplicity of the eigenvalue -1 is 1. So here we have an example of the case where the geometric multiplicity of an eigenvalue is less than its algebraic multiplicity. In the last lecture we saw an example of a matrix for which the geometric multiplicity and algebraic multiplicity of an eigenvalue were both 2. Do you think there is any relationship between the geometric and algebraic multiplicities of a multiple eigenvalue?)

(b) The characteristic polynomial is

$$\begin{aligned} p(\lambda) &= (2-\lambda)\begin{vmatrix} 5-\lambda & -3 \\ -4 & 3-\lambda \end{vmatrix} \\ &= (2-\lambda)(\lambda^2 - 8\lambda + 27) \end{aligned}$$

The characteristic polynomial cannot be factored any further (over the real numbers), so the only (real) eigenvalue of the matrix is $\lambda = 2$. A basis for the corresponding eigenspace can be found from the solution to

$$\left[\begin{array}{ccc|c} 3 & -3 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Multiplying row 1 by 1/3 gives

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Adding 4 times row 1 to row 2 gives

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Multiplying row 2 by $-1/3$ and then adding the appropriate multiple of row 2 to row 1 gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

It follows that a basis for the eigenspace is

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

You should check that the above vector is an eigenvector for the matrix with eigenvalue 2.

(c) The characteristic polynomial is

$$p(\lambda) = \begin{vmatrix} 7-\lambda & -2 & 0 \\ -2 & 6-\lambda & 2 \\ 0 & 2 & 5-\lambda \end{vmatrix}.$$

Elementary row and column operations do not help (try some to see what happens), so we just evaluate the determinant by expansion in a row or column with the most zeros. The first row will do fine:

$$\begin{aligned} p(\lambda) &= (7-\lambda)\begin{vmatrix} 6-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} + 2\begin{vmatrix} -2 & 2 \\ 0 & 5-\lambda \end{vmatrix} \\ &= (7-\lambda)((6-\lambda)(5-\lambda)-4)) + 2(-2)(5-\lambda) \\ &= (7-\lambda)(\lambda^2 - 11\lambda + 26) + 4\lambda - 20 \\ &= -\lambda^3 + 18\lambda^2 - 99\lambda + 162. \end{aligned}$$

At first glance factoring the cubic seems hopeless. However, if we can find one root of the polynomial, we can factor it to reduce the degree and then apply familiar techniques for factoring the remaining quadratic. Following the hint, take a look at problem 1(c). There it was determined that 6 is an eigenvalue for the matrix, so $\lambda - 6$ should divide the characteristic polynomial. (Ironically, expanding the determinant in the second row or column might have made that clear sooner.) In detail,

$$\begin{aligned} p(\lambda) &= -(\lambda-6)(\lambda^2 - 12\lambda + 27) \\ &= -(\lambda-6)(\lambda-3)(\lambda-9) \end{aligned}$$

so the eigenvalues are $\lambda = 3, 6, 9$. From this point on it is the usual. The basis vectors I get are

$$\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}.$$

Check!