MATH122 200610 Quiz 1C Solutions

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1. The augmented matrix corresponding to the system is

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & -1 \\ 3 & -7 & 7 & -9 \\ -4 & 6 & -1 & 4 \end{array}\right].$$

The coefficient matrix is the same as that of question 1(b) from problem set 1, so we use the same elementary row operations to transform the augmented matrix to row echelon form. Adding -3 times row 1 to row 2 and 4 times row 1 to row 3 we obtain

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & -1 \\ 0 & 2 & -5 & -6 \\ 0 & -6 & 15 & 0 \end{array}\right].$$

Adding 3 times row 2 to row 3 we obtain

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & -1 \\ 0 & 2 & -5 & -6 \\ 0 & 0 & 0 & -18 \end{array}\right].$$

The system is now in row echelon form. We could continue from this point to put the system into reduced row echelon form, but that is not necessary. We can see from the row echelon form that the system is inconsistent (has no solutions) because the final row of the augmented matrix corresponds to the equation $0 = -18$ which cannot be solved. Therefore the original system also has no solutions.

2. The system corresponds to the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 4 & 2 \\ 5 & h & k \end{array}\right].$$

We reduce it to row echelon form by adding -5 times the first row to the second to obtain

$$\left[\begin{array}{cc|c} 1 & 4 & 2 \\ 0 & h-20 & k-10 \end{array}\right].$$

   (a) The system has a unique solution if each of the columns in the coefficient matrix are pivot columns. The first column is always a pivot column, and for example, picking $h = 21$ makes the second column a pivot column. Picking $k = 11$ gives us a nice system with unique solution $x_2 = 1$, $x_1 = 2 - 4(1) = -2$. (Any other answer can be obtained by picking a value of h different from 20 and any value of k at all.)

   (b) The system has no solution if there is a row of the form $\left[\begin{array}{cc|c} 0 & 0 & \square \end{array}\right]$ where $\square$ is a non-zero number. That is only possible in our example if $h - 20 = 0$ and $k - 10 \neq 0$. So for example, picking $h = 20$ and $k = 11$ would give us a system with no solution. (Any other answer can be obtained by picking $h = 20$ and any value of k different from 10.)

   (c) The system has infinitely many solutions if there is a free variable, i.e., a column in the coefficient matrix is not a pivot column, and if the system is consistent. The only way to arrange such a situation is by taking $h = 20$ and $k = 10$.

3. Our array looks something like this:

x_1	x_2	x_3
x_4	x_5	x_6
x_7	x_8	x_9

The conditions on the variables lead to a system with augmented matrix

$$\left[\begin{array}{ccccccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{array}\right].$$

Adding −1 times row 1 to rows 4 and 7 gives

$$\left[\begin{array}{ccccccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{array}\right].$$

Swapping rows 2 and 5 gives

$$\left[\begin{array}{ccccccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{array}\right].$$

Adding 1 times row 2 to rows 4 and 7 gives

$$\left[\begin{array}{ccccccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{array}\right].$$

Swapping row 8 and row 3 gives

$$\left[\begin{array}{ccccccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right].$$

Adding 1 times row 3 to rows 4 and 7 and −1 times row 3 to row 6 gives

$$\left[\begin{array}{ccccccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right].$$

Adding −1 times row 4 to row 5 gives

$$\left[\begin{array}{ccccccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right].$$

Adding −1 times row 5 to row 6, and 3 times row 5 to row 7, gives

$$\left[\begin{array}{ccccccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & -5 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right].$$

Swapping row 6 and row 7 gives

$$\left[\begin{array}{ccccccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & -5 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right].$$

Adding −1 times row 7 to row 8 gives

$$\left[\begin{array}{ccccccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & -5 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

which is in row echelon form. Since we've gone this far, we might as well put it in reduced row echelon form. Adding 5 times row 7 to row 6, 2 times row 7 to row 5, −2 times row 7 to row 4, and −1 times row 7 to row 3 gives

$$\left[\begin{array}{ccccccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Multiplying row 6 by 1/3 gives

$$\left[\begin{array}{ccccccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Adding -1 times row 6 to row 5 gives

$$\left[\begin{array}{ccccccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Multiplying row 5 by -1 gives

$$\left[\begin{array}{ccccccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Adding -2 times row 5 to row 4, -1 times row 5 to row 3, and -1 times row 5 to row 2 gives

$$\left[\begin{array}{ccccccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Adding -1 times row 3 to row 1 gives

$$\left[\begin{array}{ccccccccc|c} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Adding -1 times row 2 to row 1 gives

$$\left[\begin{array}{ccccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The matrix is now in reduced row echelon form. We see that the final two columns in the augmented matrix are non-pivot columns, so x_8 and x_9 are free variables which we call s and t. Then

$$\begin{aligned} x_1 &= -t \\ x_2 &= -s \\ x_3 &= s + t \\ x_4 &= s + 2t \\ x_5 &= 0 \\ x_6 &= -s - 2t \\ x_7 &= -s - t \\ x_8 &= s \\ x_9 &= t. \end{aligned}$$

Putting those values back into the magic square, we have

$-t$	$-s$	$s+t$
$s+2t$	0	$-s-2t$
$-s-t$	s	t

You can easily check that the above square satisfies the given requirements for all values of the variables s and t. In particular, setting $t = 1$ and $s = 2$ and adding 5 to each cell gives the well-known '15' magic square.