MATH122 200610 Quiz 2B Solutions

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1. The equation corresponds to the augmented matrix

$$\left[\begin{array}{rrr|l} 0 & 5 & -2 & b_1 \\ -4 & 2 & -7 & b_2 \\ -8 & -11 & -8 & b_3 \end{array}\right].$$

Swapping row 1 and row 2,

$$\left[\begin{array}{rrr|l} -4 & 2 & -7 & b_2 \\ 0 & 5 & -2 & b_1 \\ -8 & -11 & -8 & b_3 \end{array}\right].$$

Adding -2 times row 1 to row 3,

$$\left[\begin{array}{rrr|l} -4 & 2 & -7 & b_2 \\ 0 & 5 & -2 & b_1 \\ 0 & -15 & 6 & -2b_2 + b_3 \end{array}\right].$$

Adding 3 times row 2 to row 3,

$$\left[\begin{array}{rrr|l} -4 & 2 & -7 & b_2 \\ 0 & 5 & -2 & b_1 \\ 0 & 0 & 0 & 3b_1 - 2b_2 + b_3 \end{array}\right].$$

The system is now in row echelon form. It does not have a solution if and only if it has a row of the form $\left[\begin{array}{ccc|c} 0 & 0 & 0 & \square \end{array}\right]$. That can be arranged, for example, by setting $b_1 = 1$, $b_2 = 0$, and $b_3 = 0$. The set of all $\mathbf{b}$ for which the system does have a solution is characterized by the equation $3b_1 - 2b_2 + b_3 = 0$.

2. The system is described by the augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 3 & 1 & -1 \\ -3 & -6 & 3 & 0 \\ 4 & 9 & -2 & -1 \end{array}\right].$$

Adding 3 times row 1 to row 2, and -4 times row 1 to row 3,

$$\left[\begin{array}{rrr|r} 1 & 3 & 1 & -1 \\ 0 & 3 & 6 & -3 \\ 0 & -3 & -6 & 3 \end{array}\right].$$

Adding 1 times row 2 to row 3,

$$\left[\begin{array}{rrr|r} 1 & 3 & 1 & -1 \\ 0 & 3 & 6 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The system is now in row echelon form. To put it into reduced row echelon form, multiply row 2 by 1/3 to obtain

$$\left[\begin{array}{rrr|r} 1 & 3 & 1 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Now add -3 times row 2 to row 1 to obtain

$$\left[\begin{array}{rrr|r} 1 & 0 & -5 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The variable x_3 is free and the solution is described by

$$x_1 = 2 + 5s$$
$$x_2 = -1 - 2s$$
$$x_3 = s,$$

or in parametric form,

$$\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} s.$$

3. Conservation of H gives the equation

$$x_1 + x_3 = 2x_6.$$

Conservation of I gives

$$x_1 + 2x_2 = x_5,$$

conservation of O gives

$$3x_1 = x_6,$$

conservation of Fe gives

$$x_2 = x_4,$$

and conservation of Cl gives

$$x_3 = 3x_4 + x_5.$$

The equations are captured by the augmented matrix

$$\left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & -2 & 0 \\ 1 & 2 & 0 & 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & -1 & 0 & 0 \end{array}\right].$$

Adding -1 times row 1 to row 2 and -3 times row 1 to row 3,

$$\left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 2 & -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & -3 & 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & -1 & 0 & 0 \end{array}\right].$$

Swapping row 2 and row 4 gives

$$\left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 5 & 0 \\ 0 & 2 & -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & -3 & -1 & 0 & 0 \end{array}\right].$$

Adding -2 times row 2 to row 4 gives

$$\left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 5 & 0 \\ 0 & 0 & -1 & 2 & -1 & 2 & 0 \\ 0 & 0 & 1 & -3 & -1 & 0 & 0 \end{array}\right].$$

Swapping row 3 and row 5 gives

$$\left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 2 & 0 \\ 0 & 0 & -3 & 0 & 0 & 5 & 0 \end{array}\right].$$

Adding 1 times row 3 to row 4 and 3 times row 3 to row 5 gives

$$\left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -2 & 2 & 0 \\ 0 & 0 & 0 & -9 & -3 & 5 & 0 \end{array}\right].$$

Adding -9 times row 4 to row 5 gives

$$\left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 15 & -13 & 0 \end{array}\right].$$

The system is in row echelon form. To put it into reduced row echelon form, multiply row 4 by -1 and row 5 by $1/15$ to obtain

$$\left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -13/15 & 0 \end{array}\right].$$

Add -2 times row 5 to row 4 and 1 times row 5 to row 3 to obtain

$$\left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 & -13/15 & 0 \\ 0 & 0 & 0 & 1 & 0 & -4/15 & 0 \\ 0 & 0 & 0 & 0 & 1 & -13/15 & 0 \end{array}\right].$$

Add 3 times row 4 to row 3, and 1 times row 4 to row 2 to obtain

$$\left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 & -4/15 & 0 \\ 0 & 0 & 1 & 0 & 0 & -25/15 & 0 \\ 0 & 0 & 0 & 1 & 0 & -4/15 & 0 \\ 0 & 0 & 0 & 0 & 1 & -13/15 & 0 \end{array}\right].$$

Finally, add -1 times row 3 to row 1 to obtain

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & -5/15 & 0 \\ 0 & 1 & 0 & 0 & 0 & -4/15 & 0 \\ 0 & 0 & 1 & 0 & 0 & -25/15 & 0 \\ 0 & 0 & 0 & 1 & 0 & -4/15 & 0 \\ 0 & 0 & 0 & 0 & 1 & -13/15 & 0 \end{array}\right].$$

The solution to the system is

$$\begin{aligned} x_1 &= \frac{1}{3}s \\ x_2 &= \frac{4}{15}s \\ x_3 &= \frac{5}{3}s \\ x_4 &= \frac{4}{15}s \\ x_5 &= \frac{13}{15}s \\ x_6 &= s. \end{aligned}$$

The smallest positive value of s which gives an all integer solution is $s = 15$ which gives $x_1 = 5$, $x_2 = 4$, $x_3 = 25$, $x_4 = 4$, $x_5 = 13$, and $x_6 = 15$. You should check that those values balance the reaction.