MATH122 200610 Quiz 3C Solutions

Edward Doolittle

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1. We must find a matrix B such that

$$\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Splitting B and O into columns, we have two conditions

$$\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The two conditions are the same, except for the subscripts, so we concentrate on solving the first. The condition is a linear system which can be solved by row reduction. The system corresponds to the augmented matrix

$$\left[\begin{array}{cc|c} 1 & -3 & 0 \\ -3 & 9 & 0 \end{array}\right].$$

Adding 3 times row 1 to row 2 gives

$$\left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The system is in reduced row echelon form. It has solution $b_{11} = 3s$, $b_{21} = s$. Taking $s = 1$, for example, we obtain the solution $b_{11} = 3$, $b_{21} = 1$. (We can't take $s = 0$ because B can't have any zero entries, but any other value for s would work.) The solution for the second column of B is exactly the same; for variety we can take $s = -1$, for example, and obtain $b_{12} = -3$, $b_{22} = -1$. This gives

$$B = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}$$

as one example of a B that works. Checking by matrix multiplication,

$$\begin{aligned} AB &= \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1(3) - 3(1) & 1(-3) - 3(-1) \\ -3(3) + 9(1) & -3(-3) + 9(-1) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

2. To find x_1, x_2, x_3 we must solve the system $B\mathbf{x} = \mathbf{0}$. Forming the augmented matrix,

$$\left[\begin{array}{ccc|c} -1 & 3 & -9 & 0 \\ 2 & 1 & 4 & 0 \\ 2 & -4 & 14 & 0 \end{array}\right].$$

Adding 2 times row 1 to row 2 and 2 times row 1 to row 3 we obtain

$$\left[\begin{array}{ccc|c} -1 & 3 & -9 & 0 \\ 0 & 7 & -14 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right].$$

Multiplying row 2 by 1/7 and row 3 by 1/3 we have

$$\left[\begin{array}{ccc|c} -1 & 3 & -9 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right].$$

Adding -1 times rows 2 to row 3 we have

$$\left[\begin{array}{ccc|c} -1 & 3 & -9 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system is now in row echelon form. Multiplying row 1 by -1 we obtain

$$\left[\begin{array}{ccc|c} 1 & -3 & 9 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Adding 3 times row 2 to row 1 we finally obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The system is in reduced row echelon form, and we can read the solution directly from it: $x_1 = -3s$, $x_2 = 2s$, $x_3 = s$. Taking $s = -1$, for example, we get the non-trivial linear relation $3\mathbf{b}_1 - 2\mathbf{b}_2 - \mathbf{b}_3 = \mathbf{0}$. (You should double check that that linear relation applies to the $\mathbf{b}_i$s.)

Now, let $C = AB$. By matrix multiplication,

$$C = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & 0 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -9 \\ 2 & 1 & 4 \\ 2 & -4 & 14 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 9 \\ -2 & -8 & 10 \\ -3 & 14 & -37 \end{bmatrix}.$$

It is straightforward to check that the same relationship as held for the $\mathbf{b}_i$s also holds for the $\mathbf{c}_i$s, i.e., $3\mathbf{c}_1 - 2\mathbf{c}_2 - \mathbf{c}_3 = \mathbf{0}$:

$$\begin{aligned} &3\mathbf{c}_1 - 2\mathbf{c}_2 - \mathbf{c}_3 \\ &= 3\begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix} - 2\begin{bmatrix} 0 \\ -8 \\ 14 \end{bmatrix} - \begin{bmatrix} 9 \\ 10 \\ -37 \end{bmatrix} \\ &= \begin{bmatrix} 9 \\ -6 \\ -9 \end{bmatrix} + \begin{bmatrix} 0 \\ 16 \\ -28 \end{bmatrix} + \begin{bmatrix} -9 \\ -10 \\ 37 \end{bmatrix} \\ &= \begin{bmatrix} 9 + 0 - 9 \\ -6 + 16 - 10 \\ -9 - 28 + 37 \end{bmatrix} = \mathbf{0}. \end{aligned}$$

(As a short cut, you know the relationship must hold because $(AB)\mathbf{x} = A(B\mathbf{x}) = A\mathbf{0} = \mathbf{0}$ by the associativity of matrix multiplication.)