MATH122 200610 Sample Final 1 Solutions DRAFT

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1. The system corresponds to the augmented matrix

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{array}\right].$$

(a) To solve the system, we perform row reduction to the augmented matrix. Adding -2 times row 1 to row 2 and -1 times row 1 to row 4 gives

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 0 & 0 & 4 & 8 & 0 & 18 & 0 \end{array}\right].$$

Adding -2 times row 2 to row 1, 5 times row 2 to row 3, and 4 times row 2 to row 4, then multiplying row 2 by -1 gives

$$\left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 6 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \end{array}\right].$$

Multiplying row 4 by $1/6$, then adding appropriate multiples of the new row 4 to the other rows, then swapping rows 3 and 4 gives

$$\left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The augmented matrix is now in reduced row echelon form. The variables $x_2 = s$ and $x_4 = t$ and $x_5 = u$ are free; the other variables are determined from the free variables. In detail, the solution is

$$\begin{aligned} x_1 &= -3s - 4t - 2u \\ x_2 &= s \\ x_3 &= -2t \\ x_4 &= t \\ x_5 &= u \\ x_6 &= 0. \end{aligned}$$

(b) In parametric form the solution can be written

$$\mathbf{x} = s\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

So the solution set is a three dimensional "hyperplane" in $\mathbb{R}^6$ parallel to the three vectors

$$\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

(I wouldn't ask for a geometric description of anything outside of $\mathbb{R}^1$, $\mathbb{R}^2$, or $\mathbb{R}^3$.)

(c) The pivot columns correspond to the non-free variables, i.e., columns 1, 3, and 6 of the coefficient matrix. A basis for the column space of the original matrix is given by those columns of the original matrix, i.e., is

$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 15 \\ 18 \end{bmatrix}.$$

As a check, you should verify that the above vectors are linearly independent, and that the other three columns can be represented as linear combinations of the above.

2. (a) We consider the augmented matrix

$$\left[\begin{array}{rrrr|l} 0 & 1 & 3 & -2 & b_1 \\ 2 & 1 & -4 & 3 & b_2 \\ 2 & 3 & 2 & -1 & b_3 \\ -4 & -3 & 5 & -4 & b_4 \end{array}\right].$$

Adding -1 times row 2 to row 3, 2 times row 2 to row 4, and then swapping rows 1 and 2, we have

$$\left[\begin{array}{rrrr|l} 2 & 1 & -4 & 3 & b_2 \\ 0 & 1 & 3 & -2 & b_1 \\ 0 & 2 & 6 & -4 & b_3 - b_2 \\ 0 & -1 & -3 & 2 & b_4 + 2b_2 \end{array}\right].$$

Adding -2 times row 2 to row 3 and 1 times row 2 to row 4 gives

$$\left[\begin{array}{rrrr|l} 2 & 1 & -4 & 3 & b_2 \\ 0 & 1 & 3 & -2 & b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_4 + 2b_2 + b_1 \end{array}\right].$$

In order for the system to be consistent we must have

$$\begin{aligned} b_3 - b_2 - 2b_1 &= 0 \\ b_4 + 2b_2 + b_1 &= 0 \end{aligned}$$

which describes the set of $\mathbf{b}$ for which the system $A\mathbf{x} = \mathbf{b}$ has a solution. Alternatively, you could note that set of $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has a solution is the column space of A, which can be described by finding a basis for the column space. Do so, and compare with the answer given above.

(b) The rank of A is the dimension of the column space; i.e., it is the number of pivot columns in A, which is 2 in this case.

(c) The nullity of A is the dimension of the null space of A, which is equal to the number of free variables in the system $A\mathbf{x} = \mathbf{0}$, which is 2 in this case. Alternatively, the rank of A plus the nullity of A is the number of columns of A by rank theorem, which again tells us that the nullity of A is 2.

(d) The kernel of a linear transformation is a concept we did not study in detail this term, but it just translates to the null space of A, which can be obtained from the reduced row echelon form of $\left[\begin{array}{c|c} A & \mathbf{0} \end{array}\right]$. Taking the above matrix in row echelon form, setting all the bs to 0, and adding -1 times row 2 to row 1 gives

$$\left[\begin{array}{rrrr|r} 2 & 0 & -7 & 5 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore in parametric form the null space of A (and hence the kernel of T) is given by

$$\mathbf{x} = s\begin{bmatrix} 7/2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

which is a plane in $\mathbb{R}^4$ parallel to the vectors

$$\begin{bmatrix} 7/2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

(e) Again, the range of a linear transformation is a concept we did not cover this term, but the concept is essentially the same as the column space of the standard matrix, which is the span of

$$\begin{bmatrix} 0 \\ 2 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ -3 \end{bmatrix},$$

i.e., a plane in $\mathbb{R}^4$ parallel to those two vectors.

3. (a) We need to determine whether the system $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution where A is the matrix formed by juxtaposing the $\mathbf{v}$s. We can either use row reduction, or because the matrix is square, we can use the Invertible Matrix Theorem (e.g., determinants, whatever). Row reduction is efficient and general purpose, so we go that way. The augmented matrix is

$$\left[\begin{array}{rrr|r} 2 & 6 & 2 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 4 & 4 & 0 \end{array}\right].$$

Adding 1 times row 1 to row 2 and multiplying row 3 by $1/4$,

$$\left[\begin{array}{rrr|r} 2 & 6 & 2 & 0 \\ 0 & 7 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right].$$

Adding -7 times row 3 to row 2 and then swapping row 2 and row 3,

$$\left[\begin{array}{ccc|c} 2 & 6 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -5 & 0 \end{array}\right].$$

The matrix is in row echelon form, from which we can see that there are no free variables, i.e. the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$, and we can conclude that the $\mathbf{v}$s are linearly independent.

(b) We did not study this. Gram-Schmidt orthogonalization will be covered in the next linear algebra course.

(c) Again, we did not study this, but we can translate the processes required into the language of dot products. We find the projection of $\mathbf{v}$ onto $\mathbf{v}_1$:

$$\operatorname{proj}_{\mathbf{v}_1} \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

That vector is in $\operatorname{span}(\mathbf{v}_1)$. Subtracting that vector from $\mathbf{v}$ gives

$$\operatorname{orth}_{\mathbf{v}_1} \mathbf{v} = \mathbf{v} - \operatorname{proj}_{\mathbf{v}_1} \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then we have that $\mathbf{v}$ is the sum of the above two vectors, one of which is parallel to $\mathbf{v}_1$, and the other of which is orthogonal to $\mathbf{v}$.

4. (a) We take dot products:

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{u}_2 &= 1(2) + 0(0) + -1(2) = 0 \\ \mathbf{u}_1 \cdot \mathbf{u}_3 &= 1(0) + 0(5) + -1(0) = 0 \\ \mathbf{u}_2 \cdot \mathbf{u}_3 &= 2(0) + 0(5) + 2(0) = 0, \end{aligned}$$

so any two of the three vectors are orthogonal.

(b) You can use row reduction, but there is a shortcut: we just find the projection of $\mathbf{u}$ onto each of those vectors, and some theory (which we did not study) tells us that $\mathbf{u}$ is the sum of the projections:

$$\begin{aligned} \operatorname{proj}_{\mathbf{u}_1} \mathbf{u} &= \frac{\mathbf{u}_1 \cdot \mathbf{u}}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 = -\frac{1}{2}\mathbf{u}_1 \\ \operatorname{proj}_{\mathbf{u}_2} \mathbf{u} &= \frac{\mathbf{u}_2 \cdot \mathbf{u}}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 = \frac{3}{4}\mathbf{u}_2 \\ \operatorname{proj}_{\mathbf{u}_3} \mathbf{u} &= \frac{\mathbf{u}_3 \cdot \mathbf{u}}{\mathbf{u}_3 \cdot \mathbf{u}_3} \mathbf{u}_3 = -\frac{1}{5}\mathbf{u}_3, \end{aligned}$$

and you should check that

$$\mathbf{u} = -\frac{1}{2}\mathbf{u}_1 + \frac{3}{4}\mathbf{u}_2 - \frac{1}{5}\mathbf{u}_3$$

holds.

(c) I assume "these vectors" refers to $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$. Dividing each vector by its length, we have the unit vectors

$$\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

You should check that the above three vectors is a set of pairwise orthogonal unit vectors, which is what is meant by an "orthonormal set" in the language of chapter 6 (which we did not study).

5. (a) Expanding $\det(A - \lambda I)$ in the second column the characteristic polynomial of the matrix is

$$\begin{aligned} p(\lambda) &= \begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda)((4-\lambda)(1-\lambda) + 2) \\ &= (1-\lambda)(\lambda^2 - 5\lambda + 6) \\ &= (1-\lambda)(\lambda - 2)(\lambda - 3). \end{aligned}$$

Expansion in any other row or column may lead to difficulties factoring the characteristic polynomial. But once it is factored, the eigenvalues are easy to find: $\lambda = 1, 2, 3$.

(b) For the eigenspace corresponding to $\lambda = 1$ we solve the system corresponding to the augmented matrix

$$\left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right].$$

You can use row reduction, but back substitution is probably a better option in situations like these. However you do it, the eigenspace is

$$\mathbf{x} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

For $\lambda = 2$ the system is

$$\left[\begin{array}{rrr|r} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{array}\right]$$

so the eigenspace is

$$\mathbf{x} = s\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}.$$

For $\lambda = 3$ the system is

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ -2 & -2 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right]$$

so the eigenspace is

$$\mathbf{x} = s\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

You should check the above eigenvectors by matrix multiplication.

(c) If $\mathbf{v}$ is an eigenvector for A with eigenvalue λ, then $\mathbf{v}$ is also an eigenvector for A^2 because

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda A\mathbf{v} = \lambda^2\mathbf{v};$$

the corresponding eigenvalue is λ^2. The reasoning can be continued for any power of A, so the eigenvalues of A^9 are $1^9 = 1$, $2^9 = 512$, and 3^9.

6. We didn't study the concept of composition of transformations in detail, but the question is essentially asking for the matrix product BA which is

$$\begin{bmatrix} 11 & 2 & 23 \\ -1 & 2 & 4 \\ 7 & 13 & 22 \\ -2 & 10 & 9 \end{bmatrix}.$$

7. As it stands, the answer can't be determined from the given data. Instead I would say, "There is a set of three column vectors which is linearly independent, and no larger set of column vectors is linearly independent." Then we can conclude that the dimension of the column space is 3. The rank theorem says that the sum of the rank plus the nullity of A is the number of columns of A, i.e. $3+$ nullity $= 8$, so the nullity is 5.

8. (a) $\det(A^2B) = \det(A)^2\det(B) = 9(-12) = -108$.

   (b) $\det(4A^{-1}) = \det(4I)\det(A)^{-1} = 4^4/3$.

   (c) $\det(AB^T) = \det(A)\det(B^T) = \det(A)\det(B) = -36$.

9. The cofactor C_{ij} is $(-1)^{i+j}M_{ij}$ where M_{ij} is the minor corresponding to ij, i.e., the determinant of the matrix obtained by deleting row i and column j.

   (a) Applying the appropriate elementary column operation, the minor is

$$M_{24} = \begin{vmatrix} 4 & -1 & 1 \\ 4 & 1 & 0 \\ 4 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 8 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{vmatrix} = 24$$

   so the cofactor is $C_{24} = (-1)^{2+4}24 = 24$.

   (b) Expanding in the second row, the minor is

$$M_{34} = \begin{vmatrix} 4 & -1 & 1 \\ 0 & 0 & -3 \\ 4 & 1 & 3 \end{vmatrix} = 3(4+4) = 24$$

   so the cofactor is $C_{34} = (-1)^{3+4}24 = -24$.

10. (a) If $\lambda = 0$ is an eigenvalue for A, then 0 is a root of the characteristic polynomial, i.e., $\det(A - 0I) = 0$. That implies that $\det(A) = 0$, so by the Invertible Matrix Theorem, A is not invertible.

   (b) If A is not invertible, then by the Invertible Matrix Theorem, $\det(A) = 0$. That implies that the characteristic polynomial $\det(A - \lambda I) = 0$ for $\lambda = 0$, i.e., 0 is a root of the characteristic polynomial.