MATH122 200610 Sample Final 1

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The following final is from 2004-20, taught by Dr. M. Torres. The course content is similar, except that we have replaced inner product spaces (questions 3(b), 3(c), and 4 below) with the Stewart handout on geometric vectors in $\mathbb{R}^3$.

1. (8 marks) Consider the following homogeneous system of linear equations

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \phantom{{}-2x_4} + 2x_5 \phantom{{}-3x_6} &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 + 10x_4 + 15x_6 &= 0 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned}$$

   (a) Find the general solution set of this system.

   (b) Give a geometric description of its general solution.

   (c) Give a basis of the column space of the coefficient matrix of the system.

2. (12 marks) Consider the following matrix.

$$A = \begin{bmatrix} 0 & 1 & 3 & -2 \\ 2 & 1 & -4 & 3 \\ 2 & 3 & 2 & -1 \\ -4 & -3 & 5 & -4 \end{bmatrix}$$

   (a) Indicate for which vectors $\mathbf{b}$ the matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution.

   (b) What is the rank of A?

   (c) What is the nullity of A?

   (d) Give a geometric description of the kernel of the linear transformation T given by $T(\mathbf{x}) = A\mathbf{x}$.

   (e) Give a geometric description of the range of the linear transformation T given by $T(\mathbf{x}) = A\mathbf{x}$.

3. (10 marks)

   (a) Indicate whether the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 1 \\ 4 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$$

   are linearly independent.

   (b) Use the Gram-Schmidt process to form an orthogonal set of vectors from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

   (c) Express the vector

$$\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

   as a sum of two vectors, one in $\text{span}(\mathbf{v}_1)$ and the other in $\text{span}(\mathbf{v}_1)^{\perp}$.

4. (8 marks)

   (a) Show that the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

   are orthogonal.

   (b) Express the vector

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

   as a linear combination of these vectors.

   (c) Normalize these vectors so that they are orthonormal.

5. (10 marks)

   (a) Find the eigenvalues of the following matrix A.

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$

   (b) Find the eigenspace corresponding to each eigenvalue.

   (c) What are the eigenvalues of A^9?

6. (4 marks) Give the standard matrix of the composition $R \circ T$ where $T(\mathbf{x}) = A\mathbf{x}$ and $R(\mathbf{x}) = B\mathbf{x}$,

$$A = \begin{bmatrix} 0 & 3 & 4 \\ 2 & -1 & 3 \\ 1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 5 & 1 \\ 1 & 0 & -1 \\ 4 & 2 & 3 \\ 3 & -1 & 0 \end{bmatrix}.$$

7. (4 marks) If A is a 5×8 matrix and three of its column vectors are linearly independent, what is the nullity of A? Explain.

8. (6 marks) Let A and B be two 4×4 matrices. Given that $\det(A) = 3$ and $\det(B) = -12$, answer the following questions.

   (a) What is $\det(A^2B)$?

   (b) What is $\det(4A^{-1})$?

   (c) What is $\det(AB^T)$?

9. (8 marks) Let A be the matrix

$$\begin{bmatrix} 4 & -1 & 1 & 6 \\ 0 & 0 & -3 & 3 \\ 4 & 1 & 0 & 14 \\ 4 & 1 & 3 & 2 \end{bmatrix}.$$

   Find the following cofactors of A.

   (a) C_{24}

   (b) C_{34}

10. (8 marks)

   (a) Show that if $\lambda = 0$ is an eigenvalue of a matrix A, then A is not invertible.

   (b) Show that if A is not invertible, then 0 is an eigenvalue of A.