# MATH122 200610 Sample Final 2 Solutions DRAFT

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1. This is a straightforward row reduction problem on the augmented matrix

$$\left[\begin{array}{rrrrr|r} 1 & -1 & 1 & -2 & 4 & 1 \\ 2 & -2 & 3 & -4 & 7 & 4 \\ -1 & 1 & -1 & 3 & -6 & 0 \\ 3 & -3 & 3 & -4 & 8 & 5 \end{array}\right].$$

Adding $-2$ times row 1 to row 2, 1 times row 1 to row 3, and $-3$ times row 1 to row 4 gives

$$\left[\begin{array}{rrrrr|r} 1 & -1 & 1 & -2 & 4 & 1 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 2 & -4 & 2 \end{array}\right].$$

Adding $-2$ times row 3 to row 4 gives

$$\left[\begin{array}{rrrrr|r} 1 & -1 & 1 & -2 & 4 & 1 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The system is now in row echelon form, from which we can see that it is solvable. To find the solution, we continue to reduced row echelon form. Adding 2 times row 3 to row 1 gives

$$\left[\begin{array}{rrrrr|r} 1 & -1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Adding $-1$ times row 2 to row 1 gives

$$\left[\begin{array}{rrrrr|r} 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We see that variables $x_2 = s$ and $x_5 = t$ are free, and the others are determined. The solution is

$$\begin{aligned} x_1 &= 1 + s - t \\ x_2 &= s \\ x_3 &= 2 + t \\ x_4 &= 1 + 2t \\ x_5 &= t, \end{aligned}$$

or in vector parametric form,

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

You should check the solution by substitution into the original equations. You can speed up the checking by showing that the vectors associated with $s$ and $t$ above are in the null space of the coefficient matrix (see 2(c) below for the method) and that the vector $\mathbf{x}$ obtained by setting $s$ and $t$ equal to zero (i.e., the first vector above) is a particular solution to the system.

2. In order to determine whether $T$ is one-to-one or onto, we must row reduce $A$ into row echelon form. Start with the matrix $A$:

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 3 & -3 & -2 & 5 \\ -2 & 2 & 0 & -5 \end{bmatrix}.$$

Adding $-3$ times row 1 to row 2 and 2 times row 1 to row 3 gives

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & -3 \end{bmatrix}.$$

Adding 2 times row 2 to row 3 gives

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The system is now in row echelon form, which is adequate to answer the next two questions.

(a) $T$ is onto if every row of $A$ is a pivot row. Since entries 11, 23 and 34 are pivot entries, each row of the row echelon form is a pivot row. It follows that each row of $A$ is a pivot row, so $T$ is onto.

(b) $T$ is one-to-one if every column of $A$ is a pivot column. In this case, column 2 of the row echelon form is not a pivot column, so column 2 of $A$ is not a pivot column, so $T$ is not one-to-one.

(c) Recall that $\mathrm{Nul}(A)$ means the null space of $A$. There are two ways to answer this question, the easy way and the hard way. Let's try the hard way first. In order to describe the null space, we solve the system $\left[\begin{array}{c|c} A & \mathbf{0} \end{array}\right]$. We're almost there: all the row operations we've done already apply to the augmented system because te column of all zeros is unchanged by the row operations. Continuing, we add $-2$ times row 3 to row 2, and

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Now we add 1 times row 2 to row 1 to obtain

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

The solution to the system in parametric form is

$$\mathbf{x} = s\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

We see that $\mathbf{u}$ is not of that form (because, for example, every vector in the null space has a 0 in the fourth entry; in general, we would need to determine whether $\mathbf{u}$ is in the span of the basis vectors of the null space which we can do by row reduction, but in this case it's easy to see whether a vector is in the null space).

Now for the easy way: $\mathbf{u}$ is in the null space if and only if $T(\mathbf{u}) = \mathbf{0}$, i.e., if and only if $A\mathbf{u} = \mathbf{0}$. In order to check that, we just do the matrix multiplication

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 3 & -3 & -2 & 5 \\ -2 & 2 & 0 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ * \\ * \\ * \end{bmatrix}$$

where we can stop when we see a non-zero entry in the product. Since there is a non-zero entry, we conclude that $\mathbf{u}$ is not in $\mathrm{Nul}(A)$. As you can see, given a single vector, it's easy to determine whether it's in the null space or not. Describing all the vectors in the null space is harder, and requires row reduction.

3. (a) There aren't a lot of 0s in the matrix, so Cramer's rule will probably be hard to apply. Instead, we use row reduction on the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 1 & 0 \\ -1 & -3 & 2 & 0 & 0 & 1 \end{array}\right].$$

Adding $-2$ times row 1 to row 2 and 1 times row 1 to row 3 gives

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 1 \end{array}\right].$$

Adding 1 times row 2 to row 3 gives

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right].$$

Adding $-2$ times row 3 to row 2, and 3 times row 3 to row 1, gives

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -2 & 3 & 3 \\ 0 & 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right].$$

Finally, adding $-2$ times row 2 to row 1 gives

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 5 & 7 \\ 0 & 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right].$$

So the inverse of $A$ is

$$A^{-1} = \begin{bmatrix} -2 & 5 & 7 \\ 0 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}.$$

You should check the answer by matrix multiplication.

(b) It's easiest to use the canned version of Cramer's rule for $2 \times 2$ matrices:

$$B^{-1} = \frac{1}{|B|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 9 & -2 \end{bmatrix}.$$

As usual, you should check by matrix multiplication.

4. (a) We could perform row/column reduction on $A$, but we can save some work by expanding in the third column, then doing some row/column reduction, then expanding again. In detail, the determinant of $A$ is

$$-2\begin{vmatrix} 2 & -1 & 3 \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{vmatrix} = -2\begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 0 & 1 & 2 \end{vmatrix}$$
$$= -2(2)(4(2) - (-2)(1)) = -4(10)$$
$$= -40.$$

You can check by evaluating the determinant in a different manner.

(b) Let's do partial row reduction, avoiding swapping rows (which can introduce factors of $-1$ which can easily be mistakenly left out). Adding $-2$ times row 2 to row 1 and 1 times row 2 to row 3, and then expanding in the first column, we have

$$|B| = \begin{vmatrix} 0 & -5 & -5 \\ 1 & 3 & 2 \\ 0 & 1 & 3 \end{vmatrix} = -\begin{vmatrix} -5 & -5 \\ 1 & 3 \end{vmatrix}$$
$$= 10.$$

You can check by evaluating the determinant in a different manner.

5. (a) We find $A^2$ by matrix multiplication:

$$A^2 = \begin{bmatrix} -1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -1 & 1 \\ -1 & 4 & -1 \\ 4 & -1 & 1 \end{bmatrix}.$$

You can check your answer by picking some vectors in $\mathbb{R}^3$, multiplying them by $A$ twice, multiplying by $A^2$ once, and comparing. If the results are different, there is a mistake somewhere.

(b) You can either work out $(B^T A)B$ or $B^T(AB)$ because matrix multiplication is associative. Let's do it the former way, the way we read:

$$\begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 1 \\ 5 & -3 \end{bmatrix}.$$

Again, you can check by picking some vectors in $\mathbb{R}^2$ and comparing the effect of multiplying by $B$, then by $A$, then by $B^T$ with the effect of multiplying by the above $2 \times 2$ matrix. If the answers are different, you made a mistake somewhere.

(c) You can't just write

$$\det(B^T B) = \det(B^T)\det(B) = (\det(B))^2$$

because $B$ is not square. You must find $B^T B$ first by matrix multiplication:

$$B^T B = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix},$$

so $\det(B^T B) = 5(2) - (-2)(-2) = 6$.

6. (a) Row reducing to row echelon form we obtain that $A$ is row equivalent to

$$\begin{bmatrix} 1 & 3 & -1 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So the pivot columns of $A$ are columns 1 and 3, so a basis for $\mathrm{Col}(A)$ is

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}.$$

You should check that those vectors are linearly independent and that the other columns are linear combinations of those columns.

(b) To find a basis for $\mathrm{Nul}(A)$ we must augment with a column of 0s and continue to reduced row echelon form. Adding 1 times row 2 to row 1 gives

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We have $x_2 = s$ and $x_4 = t$ free, and the parametric form of the solution space is

$$\mathbf{x} = s\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

and it follows that a basis for the null space is

$$\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

You should check that those vectors are linearly independent and are in the null space. (They also help express the non-basis columns of $A$ in terms of the basis columns, so help with checking in (a).)

(c) The rank of $A$ is the dimension of the column space, which is just the number of elements in a basis, which is 2.

7. (a) The characteristic polynomial is

$$p(\lambda) = \begin{vmatrix} 4-\lambda & -1 \\ 4 & -2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 4.$$

The characteristic polynomial can't be factored easily in this case so we use the quadratic formula:

$$\lambda = \frac{2 \pm \sqrt{4-4(-4)}}{2} = 1 \pm \sqrt{5}.$$

(b) For each eigenvalue $\lambda$, we must find the null space of $A - \lambda I$. Let's start with $\lambda = 1 - \sqrt{5}$. Then the augmented matrix we need to solve is

$$\left[\begin{array}{cc|c} 3+\sqrt{5} & -1 & 0 \\ 4 & -3+\sqrt{5} & 0 \end{array}\right].$$

Multiplying row 1 by 4 and row 2 by $3+\sqrt{5}$ gives

$$\left[\begin{array}{cc|c} 4(3+\sqrt{5}) & -4 & 0 \\ 4(3+\sqrt{5}) & -4 & 0 \end{array}\right]$$

where we have used $(3+\sqrt{5})(-3+\sqrt{5}) = 5 - 9 = -4$. Therefore a basis for the eigenspace corresponding to $\lambda = 1 - \sqrt{5}$ is

$$\begin{bmatrix} 1 \\ 3+\sqrt{5} \end{bmatrix}.$$

(Check by matrix multiplication.) Similarly, to find the eigenspace corresponding to the eigenvalue $1+\sqrt{5}$ we must solve the system

$$\left[\begin{array}{cc|c} 3-\sqrt{5} & -1 & 0 \\ 4 & -3-\sqrt{5} & 0 \end{array}\right].$$

Multiplying row 1 by 4 and row 2 by $3-\sqrt{5}$ gives

$$\left[\begin{array}{cc|c} 4(3-\sqrt{5}) & -4 & 0 \\ 4(3-\sqrt{5}) & -4 & 0 \end{array}\right]$$

where we have again used $(3+\sqrt{5})(-3+\sqrt{5}) = 5 - 9 = -4$. Therefore a basis for the eigenspace corresponding to $\lambda = 1+\sqrt{5}$ is

$$\begin{bmatrix} 1 \\ 3-\sqrt{5} \end{bmatrix}.$$

(Check by matrix multiplication.)

(c) The eigenvalues of $A^{-1}$ can be found by calculating $A^{-1}$ and then finding the eigenvalues, but there is a short cut. Suppose $\mathbf{v}$ is an eigenvector of $A$ with eigenvalue $\lambda$. Then $\mathbf{v} = I\mathbf{v} = A^{-1}A\mathbf{v} = A^{-1}\lambda\mathbf{v}$, so $A^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}$, so $\mathbf{v}$ is an eigenvector of $A^{-1}$ with eigenvalue $\lambda^{-1}$. It follows that the eigenvalues of $A^{-1}$ are $(1-\sqrt{5})^{-1}$ and $(1+\sqrt{5})^{-1}$. You should be able to find representation for those numbers in the form $p + q\sqrt{5}$ where $p$ and $q$ are rational numbers.

In the above argument we implicitly made use of the fact that $\lambda \neq 0$. Why can we assume that is true?

(d) A similar argument applies for positive powers of $A$. See the solution of 10(d) for a sketch of the argument. So the eigenvalues of $A^5$ are $(1-\sqrt{5})^5$ and $(1+\sqrt{5})^5$. It's harder to find an expression for those numbers in the form $p + q\sqrt{5}$, so don't worry about it.

8. This question uses language and concepts we did not study, in particular “orthogonal basis” and Gram-Schmidt orthogonalization. In terms of our approach to the material, you would be asked to find a vector $\mathbf{v}_2'$ which is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ and which is orthogonal to $\mathbf{v}_1$. You can answer (b) using row reduction; it would be best to use $\mathbf{v}_1$ and $\mathbf{v}_2'$ as the basis of $V$. You can then answer (c) using the numerical answer in (b). In detail,

   (a) Take the projection of $\mathbf{v}_2$ onto $\mathbf{v}_1$:

   $$\operatorname{proj}_{\mathbf{v}_1} \mathbf{v}_2 = \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \frac{8}{9}\mathbf{v}_1.$$

   Subtracting the above from $\mathbf{v}_2$ gives a vector which is both a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, and which is orthogonal to $\mathbf{v}_1$:

   $$\mathbf{v}_2' = \mathbf{v}_2 - \operatorname{proj}_{\mathbf{v}_1} \mathbf{v}_2 = \begin{bmatrix} -7/9 \\ -10/9 \\ 2/9 \end{bmatrix}.$$

   It should be clear that the above vector is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, so it is in $V$, and it is linearly independent with $\mathbf{v}_1$, so $\mathbf{v}_1$ and $\mathbf{v}_2'$ together form a basis for $V$. Furthermore, $\mathbf{v}_2'$ is orthogonal to $\mathbf{v}_1$ as you can check by dot products. So we have found an orthogonal basis. (This is the essence of Gram-Schmidt orthogonalization.) We can replace $\mathbf{v}_2'$ with any nonzero multiple and preserve these properties, so let’s multiply by 9 in what follows.

   (b) We can answer this question by row reduction. (We could also check whether $\mathbf{v} = \operatorname{proj}_{\mathbf{v}_1} \mathbf{v} + \operatorname{proj}_{\mathbf{v}_2'} \mathbf{v}$, but we didn’t learn that.) In detail, we must check whether the system

   $$\left[\begin{array}{cc|c} 2 & -7 & 1 \\ -1 & -10 & 4 \\ 2 & 2 & -2 \end{array}\right]$$

   has a solution. Adding 2 times row 2 to rows 1 and 3, multiplying row 2 by $-1$, and then swapping rows 1 and 2 gives

   $$\left[\begin{array}{cc|c} 1 & 10 & -4 \\ 0 & -27 & 9 \\ 0 & -18 & 6 \end{array}\right].$$

   Multiplying row 2 by $-1/9$ and row 3 by $-1/6$, and row 1 by 3 gives

   $$\left[\begin{array}{cc|c} 3 & 30 & -12 \\ 0 & 3 & -1 \\ 0 & 3 & -1 \end{array}\right].$$

   Adding $-1$ times row 2 to row 3 and $-10$ times row 2 to row 1 gives

   $$\left[\begin{array}{cc|c} 3 & 0 & -2 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{array}\right].$$

   It follows that

   $$\mathbf{v} = -\frac{2}{3}\mathbf{v}_1 - \frac{1}{3}\mathbf{v}_2'$$

   which you should check. So $\mathbf{v}$ is in $V$.

   (c) The $B$ coordinates of $\mathbf{v}$ is just a fancy way of writing down the answer we found in (b):

   $$[\mathbf{v}]_B = \begin{bmatrix} -2/3 \\ -1/3 \end{bmatrix}.$$

9. This question is outside our domain. You will learn this material if you take the next linear algebra course.

10. I will give explanations for your benefit, although none was required.

   (a) False; $A$ could be the matrix of all zeros, for example, in which case $T$ would definitely not be one-to-one.

   (b) False; again $A$ could be the matrix of all zeros, and $B$ and $C$ could be different invertible matrices.

   (c) True; in order for $T$ to be onto, every row of $A$ must be a pivot row, but it’s only possible to have at most three pivot entries in $A$ because there are three columns. Therefore there can be at most three pivot rows, and $T$ cannot be onto.

   (d) True; if $\lambda$ is an eigenvalue for $A$, that means there is a non-zero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. Then we have $A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v}$, so $\lambda^2$ is an eigenvalue of $A^2$. The above calculation depends on the fact that $A\lambda = \lambda A$ for any matrix $A$.

   (e) False; one of the eigenvalues could be zero, in which case $A$ would not be invertible. Consider, for example, the matrix with the entries 0, 1, and 2 along the diagonal and zeros elsewhere. It is not invertible (because the determinant is $0 \cdot 1 \cdot 2 = 0$, for example), but it has three distinct eigenvalues.