MATH122 200610 Sample Final 2

Edward Doolittle

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The following final is from 2005-10, taught by Dr. S. Panafidin. The course content is similar, except that we have replaced inner product spaces (questions 8(a) and 9 below) with the Stewart handout on geometric vectors in $\mathbb{R}^3$.

1. (10 marks) Solve the following system of linear equations. Write the solution in parametric vector form.

$$\begin{aligned} x_1 - x_2 + x_3 - 2x_4 + 4x_5 &= 1 \\ 2x_1 - 2x_2 + 3x_3 - 4x_4 + 7x_5 &= 4 \\ -x_1 + x_2 - x_3 + 3x_4 - 6x_5 &= 0 \\ 3x_1 - 3x_2 + 3x_3 - 4x_4 + 8x_5 &= 5 \end{aligned}$$

2. (10 marks) For the matrix

$$A = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 3 & -3 & -2 & 5 \\ -2 & 2 & 0 & -5 \end{bmatrix}$$

and the linear transformation T defined by $T(\mathbf{x}) = A\mathbf{x}$, answer the following questions.

   (a) Is T one-to-one?

   (b) Is T onto?

   (c) Is the vector

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}$$

   in $\mathrm{Nul}(A)$?

3. (10 marks) Find the inverse of each of the following matrices.

   (a) $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -4 \\ -1 & -3 & 2 \end{bmatrix}$

   (b) $B = \begin{bmatrix} 2 & 1 \\ 9 & 4 \end{bmatrix}$

4. (10 marks) Compute the determinant of the given matrix.

   (a) $A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & 2 & -1 \\ 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

   (b) $B = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & -2 & 1 \end{bmatrix}$

5. (10 marks) For the matrices

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(a) Compute A^2.
(b) Compute $B^T AB$.
(c) Compute $\det(B^T B)$.

6. (10 marks) For the matrix

$$A = \begin{bmatrix} 1 & 3 & -1 & 4 \\ 2 & 6 & -1 & 6 \\ -1 & -3 & -1 & 0 \end{bmatrix}$$

(a) Find a basis of $\mathrm{Col}(A)$.
(b) Find a basis of $\mathrm{Nul}(A)$.
(c) Find $\mathrm{rank}(A)$.

7. (10 marks) For the matrix

$$A = \begin{bmatrix} 4 & -1 \\ 4 & -2 \end{bmatrix}$$

(a) Find the eigenvalues of A.
(b) For each eigenvalue, find the corresponding eigenspace.
(c) Find the eigenvalues of A^{-1}.
(d) Find the eigenvalues of A^5.

8. (10 marks) Consider the vector space V spanned by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}.$$

(a) Find an orthogonal basis of V.
(b) Is the vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$$

in V?
(c) Find the coordinates of $\mathbf{v}$ relative to the basis that you found in part (a).

9. (10 marks) The subspace W of $\mathbb{R}^4$ has the basis

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

(a) Find the orthogonal projection of the vector

$$\mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

on W.

(b) Find the orthogonal projection of $\mathbf{w}$ on $W^{\perp}$.

10. (10 marks) Mark each of the following statements as true or false (no explanation necessary).

   (a) If A is a 5×3 matrix then the linear transformation T with standard matrix A must be one-to-one.

   (b) For square matrices A, B, and C, if B and C are invertible and $AB = AC$ then $B = C$.

   (c) If A is a 4×3 matrix then the linear transformation T with standard matrix A cannot be onto.

   (d) If a is an eigenvalue of A then a^2 is an eigenvalue of A^2.

   (e) If the 3×3 matrix A has 3 distinct eigenvalues then A must be invertible.