MATH221-001 200530 Practice Midterm Test 2 Solutions DRAFT

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1. Refer to the truth table in Table 1. The columns for $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are the same so the two

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
F	F	F	T	T	T	T
F	T	F	T	T	F	T
T	F	F	T	F	T	T
T	T	T	F	F	F	F

Table 1: Truth table for $\neg(p \wedge q)$ and $\neg p \vee \neg q$

expressions are logically equivalent.

2. By the previous result,

$$\begin{aligned} x \in (A \cap B)' &\Leftrightarrow \neg(x \in (A \cap B) \\ &\Leftrightarrow \neg(x \in A \wedge x \in B) \\ &\Leftrightarrow \neg(x \in A) \vee \neg(x \in B) \\ &\Leftrightarrow (x \in A') \vee (x \in B') \\ &\Leftrightarrow x \in (A' \cup B'). \end{aligned}$$

An element x of the universal set X is in $(A \cap B)'$ if and only if it is in $A' \cup B'$, so the two sets are equal.

3. The statement q is "n is not a multiple of 7" and the statement r is "n is a multiple of 3". The contrapositive, converse, and contrary of p are in Table 2. Note that the double negative rule $\neg(\neg s) \equiv s$

	In symbols	In words
Statement	$q \Rightarrow r$	If n is not a multiple of 7 then n is a multiple of 3.
Contrapositive	$\neg r \Rightarrow \neg q$	If n is not a multiple of 3 then n is a multiple of 7.
Converse	$r \Rightarrow q$	If n is a multiple of 3 then n is not a multiple of 7.
Contrary	$\neg q \Rightarrow \neg r$	If n is a multiple of 7 then n is not a multiple of 3.

Table 2: Contrapositive, converse, and contrary of a statement

has been used.

4. By commutativity of multiplication,

$$(b + c) \times a = a \times (b + c).$$

By distributivity,

$$a \times (b + c) = (a \times b) + (a \times c).$$

By commutativity of multiplication applied to each of the bracketed terms above,

$$(a \times b) + (a \times c) = (b \times a) + (c \times a).$$

Chaining the above three equations together gives the required result.

5. (a) The base case is $P(1)$, i.e., "$1^3 + 5 \times 1$ is a multiple of 6", which is true because 6 is certainly a multiple of 6. (If we're being really picky, $6 = 6 \times 1$. But we're not being that picky.)

   (b) The induction hypothesis is $P(k)$ for some particular k, i.e., "$k^3 + 5k$ is a multiple of 6". It might be best to rewrite the induction hypothesis in the form of an equation: $k^3 + 5k = 6p$ for some $p \in \mathbb{N}$.

   (c) The induction step is $P(k) \Rightarrow P(k+1)$, i.e., "$k^3+5k = 6p$ for some $p \in \mathbb{N}$ implies $(k+1)^3+5(k+1) = 6q$ for some $q \in \mathbb{N}$". To prove the induction step, start with the left hand side of $P(k+1)$ and transform it into the right hand side, using the induction hypothesis where necessary:

   $$(k+1)^3 + 5(k+1) = k^3 + 3k^2 + 3k + 1 + 5k + 5 = k^3 + 5k + 3k^2 + 3k + 6 = 6p + 6 + 3k(k+1).$$

   Since either 2 divides k or 2 divides $k+1$, 2 always divides $k(k+1)$; we can write $k(k+1) = 2r$ for some $r \in \mathbb{N}$. Then

   $$(k+1)^3 + 5(k+1) = 6p + 6 + 6r = 6(p + r + 1)$$

   is a multiple of 6 which establishes the induction step.

6. We prove the result by induction. The base case is $\sum_{i=1}^{1} i(i+1) = \frac{1(1+1)(1+2)}{3}$ which is true because the left hand side is $1(1+1) = 2$ and the right hand side is $\frac{1(1+1)(1+2)}{3} = \frac{6}{3} = 2$, and the two sides are equal. Assuming the induction hypothesis $\sum_{i=1}^{k} i(i+1) = \frac{k(k+1)(k+2)}{3}$ we have

$$\begin{aligned}
\sum_{i=1}^{k+1} i(i+1) &= \left(\sum_{i=1}^{k} i(i+1)\right) + (k+1)((k+1)+1) \\
&= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\
&= (k+1)(k+2)\left(\frac{k}{3} + 1\right) \\
&= (k+1)(k+2)\left(\frac{k+3}{3}\right) \\
&= \frac{(k+1)((k+1)+1)((k+1)+2)}{3}
\end{aligned}$$

which proves the induction step. Therefore the result is true for all $n \in \mathbb{N}$ by induction.

7. (a) Take off 9 cents until we reach a multiple of 4: 39 is not a multiple of 4, $39 - 9 = 30$ is not a multiple of 4, $39 - 18 = 21$ is not a multiple of 4, but $39 - 27 = 12$ is a multiple of 4. Therefore $39 = 4 \times 3 + 9 \times 3$.

   (b) We prove the result by strong induction. For the base case, we check the first four values starting at 36: $36 = 9 \times 4$ (four 9-cent stamps) is easy; $37 = 28+9 = 4\times7+9\times1$, $38 = 20+18 = 4\times5+9\times2$, and we have done 39 already.

   For the induction step, we assume the strong induction hypothesis that any value between 36 and k can be made up of 4 and 9-cent stamps where $k \geq 40$. Then $k+1 = k-3+4 = 4m+9n+4 = 4(m+1)+9n$ can also be made up of 4 and 9-cent stamps.

(c) The only way I can think of to do this offhand is check each value from 35 down.

$$
\begin{aligned}
35 &= 26+9 = 17+18 = 8+27 = 4\times 2 + 9\times 3 \\
34 &= 25+9 = 16+18 = 4\times 4 + 9\times 2 \\
33 &= 24+9 = 4\times 6 + 9\times 1 \\
32 &= 4\times 8 \\
31 &= 22+9 = 13+18 = 4+27 = 4\times 1 + 9\times 3 \\
30 &= 21+9 = 12+18 = 4\times 3 + 9\times 2 \\
29 &= 20+9 = 4\times 5 + 9\times 1 \\
28 &= 4\times 7 \\
27 &= 9\times 3 \\
26 &= 17+9 = 8+18 = 4\times 2 + 9\times 2 \\
25 &= 16+9 = 4\times 4 + 9\times 1 \\
24 &= 4\times 6 \\
23 &= 14+9 = 5+18
\end{aligned}
$$

so 23 is the largest number that cannot be expressed as a sum of 4's and 9's, and the smallest n_0 is therefore 24. [I should have made the initial n_0 smaller than 36 so that you wouldn't have to check so many cases during the test. But it is possible to do this check quickly.]