MATH221-001 200530 Midterm Test 2 Solutions DRAFT

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1. See Table 1 for the completion of the truth table. The spaces that are blank aren't required thanks to 'short-circuiting'. The logical expressions $\neg(p \wedge \neg q)$ and $\neg p \vee q$ are logically equivalent because the corresponding

p	q	p	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$	$\neg p$	q	$\neg p \vee q$
F	F	F		F	T	T		T
F	T	F		F	T	T		T
T	F	T	T	T	F	F	F	F
T	T	T	F	F	T	F	T	T

Table 1: Truth table for $\neg(p \wedge \neg q)$ and $\neg p \vee q$

columns in the truth table are identical.

2. By the definitions,

$$x \in (A \cap B')' \Leftrightarrow \neg(x \in A \cap B') \Leftrightarrow \neg(x \in A \wedge x \in B') \Leftrightarrow \neg(x \in A \wedge \neg(x \in B)).$$

By the above logical equivalence,

$$\neg(x \in A \wedge \neg(x \in B)) \Leftrightarrow \neg(x \in A) \vee (x \in B) \Leftrightarrow x \in A' \vee x \in B \Leftrightarrow x \in (A' \cup B)$$

Since an arbitrary element x of X is in $(A \cap B')'$ if and only if it is in $A' \cup B$, the two sets must be equal.

3. The statement q is "m^2 is a multiple of 2" and the statement r is "m is a multiple of 2". The contrapositive of p in symbols is $\neg r \Rightarrow \neg q$, which in words is "if m is not a multiple of 2 then m^2 is not a multiple of 2". The contrapositive of p can also be written "if m is odd then m^2 is odd". In that form, it can easily be proven either by algebra ($m = 2n+1$ implies $m^2 = 4n^2 + 4n + 1$ which is odd), by modular arithmetic ($m \equiv 1 \bmod 2$ implies $m^2 \equiv 1^2 \equiv 1 \bmod 2$), or by induction. Since the contrapositive of p is true, the statement p itself is true since the two are logically equivalent.

4. By the distributive law (axiom 9) applied with $a = (x+y)$, $b = x$ and $c = y$,

$$(x+y)^2 = (x+y) \times (x+y) = ((x+y) \times x) + ((x+y) \times y).$$

By the commutative law for multiplication (axiom 5) applied twice,

$$((x+y) \times x) + ((x+y) \times y) = (x \times (x+y)) + ((x+y) \times y) = (x \times (x+y)) + (y \times (x+y)).$$

Applying the distributive law (axiom 9) again, twice,

$$(x \times (x+y)) + (y \times (x+y)) = ((x \times x) + (x \times y)) + ((y \times x) + (y \times y)) = x \times x + x \times y + y \times x + y \times y$$

where the brackets can been dropped in the last expression because of order of operations and the associativity of addition (axiom 4). Applying the commutative law for multiplication (axiom 5) applied to the third term, and then the existence and properties of 1 (axiom 7),

$$x \times x + x \times y + y \times x + y \times y = x^2 + xy + xy + y^2 = x^2 + (xy)1 + (xy)1 + y^2.$$

The $\times$ symbol has been dropped above for conciseness. Now applying the distributive law (axiom 9) and the commutative law for multiplication (axiom 5) one last time,

$$x^2+(xy)1+(xy)1+y^2 = x^2+(xy)(1+1)+y^2 = x^2+(1+1)xy+y^2 = x^2+2xy+y^2$$

where 2 is defined to be $1+1$ and the bracket around xy can be dropped by the associativity of multiplication (axiom 6). Skipping a few steps here and there is OK as long as I can still find 5 marks worth.

5. Consider the statement $P(n) =$ "$3^{2n+1}+1$ is a multiple of 4". Show that $P(n)$ is true for all $n \in \mathbb{N}$ by following the steps below.

   (a) The base case is $P(1) =$ "$3^{2(1)+1}+1$ is a multiple of 4". The base case is true because $3^3+1 = 28 = 4\times 7$ is indeed a multiple of 4.

   (b) The induction hypothesis is "for some k, $3^{2k+1}+1$ is a multiple of 4". It might be better to write the induction hypothesis in the form of an equation: "for some k, $3^{2k+1}+1 = 4m$."

   (c) The induction step is to show that $P(k) \Rightarrow P(k+1)$ for any k. So, suppose $P(k)$, i.e., the induction hypothesis in the previous answer. Then $P(k+1)$ is true if $3^{2(k+1)+1}+1$ is a multiple of 4, but $3^{2k+3}+1 = 3^{2k+1}\times 9+1$. By the induction hypothesis, $3^{2k+1} = 4m-1$ so we have $3^{2k+1}\times 9+1 = 9(4m-1)+1 = 36m-9+1 = 36m-8 = 4(9m-2)$ is also a multiple of 4, which proves the induction step. It follows that $P(n)$ is true for all $n \in \mathbb{N}$.

6. The base case occurs when $n=1$. In that case the left hand side is $\sum_{i=1}^{1} i(i+2) = 1(1+2) = 3$ and the right hand side is $1(1+1)(2(1)+7)/6 = 2(9)/6 = 18/6 = 3$. The two sides are equal which proves the base case.

   To prove the induction step, assume that $\displaystyle\sum_{i=1}^{k} i(i+2) = \frac{k(k+1)(2k+7)}{6}$ is true for some k. Then

   $$\sum_{i=1}^{k+1} i(i+2) = \sum_{i=1}^{k} i(i+2) + (k+1)(k+3) = \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3)$$

   by the induction hypothesis. Then factoring out a common factor of $k+1$ and adding the two terms gives

   $$\sum_{i=1}^{k+1} i(i+2) = (k+1)\frac{2k^2+7k+6k+18}{6} = (k+1)\frac{(k+2)(2k+9)}{6} = \frac{(k+1)((k+1)+1)(2(k+1)+7)}{6}$$

   which proves the induction step. The theorem follows by induction.

7. We only need weak induction, not strong induction, to prove this result. Let $P(n)$ be the statement "f_{3n} is even". Then $P(1)$ is true because $f_{3(1)} = f_3 = 2$ is indeed even. Furthermore,

   $$f_{3(n+1)} = f_{3n+3} = f_{3n+2} + f_{3n+1} = f_{3n+1} + f_{3n} + f_{3n+1} = f_{3n} + 2f_{3n+1}.$$

   By the induction hypothesis, f_{3n} is even, and clearly $2f_{3n+1}$ is even, so $f_{3(n+1)}$ is even and the result follows by induction.

   On the other hand, you can use strong induction to prove the result if you strengthen the induction hypothesis (the inventor's paradox). Let $Q(n)$ be the statement "f_n is even if $n \equiv 0 \bmod 3$; otherwise f_n is odd". Note that this statement is stronger than what we need to prove. For the base case, you should check that $Q(1)$, $Q(2)$, and $Q(3)$ are true. As the induction hypothesis, assume that $Q(m)$ is true for all $m \le k$. Then for the induction step we must use the induction hypothesis to prove $Q(k+1)$. There are two possible cases. If $k+1 \equiv 0 \bmod 3$ then neither k nor $k-1$ are congruent to 0 mod 3 so our induction hypothesis says that both f_k and f_{k-1} are odd; but then $f_{k+1} = f_k + f_{k-1}$ is the sum of two odd numbers, so is even, so $Q(k+1)$ is true in this case. On the other hand, if $k+1 \not\equiv 0 \bmod 3$ then one of k and $k-1$ is congruent to 0 mod 3 and the other isn't, so by the induction hypothesis one of f_k and f_{k-1} is odd and the other is even, so $f_{k+1} = f_k + f_{k-1}$ is odd, so $Q(k+1)$ is true in this case as well. In all cases, $Q(k+1)$ is true, so the induction step is true, so by strong induction $Q(n)$ is true for all $n \in \mathbb{N}$.