# MATH221-001 200530 Problem Set 2 Solutions

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1. (a) Since $3726353 = 372635 \times 10 + 3$, $3726353 \equiv 3 \bmod 10$. Similarly, any number is congruent to its last digit mod 10, so we have $3726353 \times 1939678 \equiv 3 \times 8 = 24 \equiv 4 \bmod 10$.

   (b) There is a similar shortcut modulo 25. Since $4939883 = 49398 \times 4 \times 25 + 83 = 49398 \times 4 \times 25 + 3 \times 25 + 8$, $4939883 \equiv 8 \bmod 25$. Similarly, $385677 \equiv 77 \equiv 2 \bmod 25$, so $4939883 \times 385677 \equiv 8 \times 2 \equiv 16 \equiv 16 - 25 = -9 \bmod 25$.

2. (a) $3726353 \equiv 3+7+2+6+3+5+3 = 29 \equiv 2+9 = 11 \equiv 1+1 = 2 \bmod 9$. (This is faster if we throw away any 9's that we see; for example, we could have gone straight from 29 to 2 by 'casting out' the 9. We can also throw away pairs of digits that add up to 9, so the 7 and 2, and 6 and 3 could have been thrown away right at the beginning; you could have just used $3+5+3 \equiv 11 \equiv 11-9 = 2$.) Similarly $1939678 \equiv 7 \bmod 9$. Multiplying those values together, $2 \times 7 \equiv 14 \equiv 5 \bmod 9$. On the other hand, $7227924943334 \equiv 2444 \equiv 14 \equiv 5 \bmod 9$, so the result checks out. Casting out 9's doesn't prove that the result is incorrect.

   On the other hand, the result really is incorrect. The correct product is 7227924934334, as you can tell with Windows calculator or another calculator with 14 digits of accuracy. Two digits have been transposed in the result, which is one kind of error that casting out 9's can never detect. (Why not?)

   (b) Casting out 9's, $4939883 \equiv 43883 \equiv 683 \equiv 53 \equiv 8 \bmod 9$, and $385677 \equiv 8577 \equiv 477 \equiv 27 \equiv 0 \bmod 9$. Multiplying, $8 \times 0 \equiv 0 \bmod 9$. On the other hand, $1905199255691 \equiv 15125561 \equiv 5255 \equiv 755 \equiv 35 \equiv 8 \bmod 9$, so the two sides do not agree mod 9, so the result cannot be correct.

3. Since a number and the sum of its digits agree mod 9, the difference of the two is 0 mod 9, i.e., the difference must be a multiple of 9. Furthermore, since you originally thought of a two-digit number, the difference must be 81 or smaller. Notice that the symbol for all multiples of 9 less than or equal to 81 (i.e., 0, 9, 18, 27, 36, 45, 54, 63, 72, 81) is the same, so no matter what number you started out with, you would have picked the same symbol, which of course is the symbol that appears in the crystal ball.

4. Theorem: If 40,000,000 is a multiple of 375, then 120,000,000 is a multiple of 375. Proof: If 40,000,000 is a multiple of 375, then $40,000,000 = 375n$ for some integer $n$, in which case $120,000,000 = 3 \times 40,000,000 = 3 \times 375n = 375(3n)$ is also a multiple of 375.

   40,000,000 is not a multiple of 375: one way of seeing that is noting that 375 is a multiple of 3, while 40,000,000 is not. Or you could do long division and get a non-zero remainder. The fact that 40,000,000 is not a multiple of 375 is not relevant to the answer to the first part; the theorem is still true and the proof is still valid. (That is because of the fact that if the antecedent of an implication is false, then the implication itself is true. It would be possible to prove that "if 40,000,000 is a multiple of 375 then the moon is made of green cheese," for example, although the proof would be much more contrived.)

5. By the division algorithm, $329 = 29 \times 11 + 10$; on the other hand, $3 - 2 + 9 = 10$, so the two procedures agree. Similarly, $666 = 60 \times 11 + 6$ and $6 - 6 + 6 = 6$ so the two procedures again agree. However, $291 = 26 \times 11 + 5$ but $2 - 9 + 1 = -6$. The two results do not agree, but they are congruent mod 11.

   That is true in general for three digit numbers. Suppose the base 10 representation for the number is $(abc)_{10}$. (We use that special notation instead of $abc$ because the latter may be confused with $a \times b \times c$.) Then $(abc)_{10} = a \times 100 + b \times 10 + c = a \times 99 + a + b \times 11 - b + c \equiv a - b + c \bmod 11$.

In fact, following the textbook, we can find a procedure for 'casting out 11s' just as we did for casting out 9s. The important idea is that $10 \equiv -1 \bmod 11$, so $100 = 10^2 \equiv (-1)^2 = 1 \bmod 11$, $1000 = 10^3 \equiv (-1)^3 = -1 \bmod 11$, and so on. The general pattern is $10^n \equiv 1 \bmod 11$ if $n$ is even and $10^n \equiv -1 \bmod 11$ if $n$ is odd. So, for example, in base 10 notation, $(abcd)_{10} = a \times 10^3 + b \times 10^2 + c \times 10^1 + d \times 10^0 \equiv a \times -1 + b \times 1 + c \times -1 + d \times 1 = -a + b - c + d \bmod 11$. In general, we have the Theorem: $(a_n a_{n-1} \dots a_2 a_1 a_0)_{10} \equiv (-1)^n a_n + (-1)^{n-1} a_{n-1} + \cdots + (-1)^2 a_2 + (-1)^1 a_1 + (-1)^0 a_0$. Proof: $(a_n a_{n-1} \dots a_2 a_1 a_0)_{10} = a_n \times 10^n + a_{n-1} \times 10^{n-1} + \cdots + a_2 \times 10^2 + a_1 \times 10^1 + a_0 \times 10^0 \equiv (-1)^n a_n + (-1)^{n-1} a_{n-1} + \cdots + (-1)^2 a_2 + (-1)^1 a_1 + (-1)^0 a_0$. by the above calculations with $10^k$.

6. The general idea is very similar to the Mystical Ball site of question 3. Changing the order of the digits doesn't change the digit sum, so no matter how the digits are rearranged the residue mod 9 is the same. Subtracting the two numbers gives a result which is congruent to 0 mod 9. Suppose the four digits are $(abcd)_{10}$, where $a$ may be a leading zero. Since $(abcd)_{10} \equiv 0 \bmod 9$, we know that $a + b + c + d \equiv 0 \bmod 9$. Then if, say, $b$ is the digit taken out, then $b \equiv -a - c - d \bmod 9$. We just add 9 to $-a - c - d$ repeatedly until it falls in the range 1–9 to get the only possible value of the missing digit. The reason you're asked not to circle a 0 is because there would be no way of telling whether the missing digit were a 0 or a 9; since you are asked not to circle a 0, you know for sure that the answer must be 9 if $-a - c - d \equiv 0 \bmod 9$.

   Giving an example might help clarify the steps. Suppose the number initially chosen is 3278. Scramble the digits to get 8237. Subtracting the larger from the smaller gives 4959. Note that 4959 is divisible by 9 and its digit sum is also divisible by 9. Suppose you circled 5. The remaining digigs are 4, 9, and 9. $-4 - 9 - 9 = -22$. Repeatedly adding 9 to $-22$ gives in order $-13$, $-4$, 5 which is the answer. On the other hand, suppose you had circled a 9 in 4959. Then the negative of the sum of the remaining digits is $-4 - 9 - 5 = -18$. Repeatedly adding 9 gives in order $-9$, 0, and 9. You don't stop at 0 because you know that 0 can't be the missing digit because you were asked not to circle a 9.

   Note that the whole procedure falls apart if you initially pick a number with four identical digits.

7. The month offset for September is 4, the day within the month is unknown (call it $n$), the century offset for 1900's is 1, the number of years in the century is 73, and the number of leap days since the beginning of the century is $\lfloor \frac{73}{4} \rfloor = 18$. Altogether the day of the week is $4 + n + 1 + 73 + 18 = n + 96$. We want $n + 96 \equiv 1 \bmod 7$. Subtracting 96 from both sides, we get $n \equiv -96 \equiv -3 \equiv 4 \bmod 7$. All possible values of $n$ are $n = 3, 10, 17, 24$ (other values of $n$ such as $n = -4$ or $n = 31$ can satisfy the congruence $n \equiv 4 \bmod 7$ but are not possible days in the month of September). The first Monday in September, 1973, was therefore September 3, 1973.

   Alternatively, you could have guessed September 1, 1973, figured out that was the wrong day of the week, and then added days until it was a Monday.

   One extra bonus mark if you named a holiday for which the formula can help you determine the date (almost any holiday in our calendar, e.g., New Year's Day, Christmas Day, 'First Monday in August', etc.) and a holiday for which the formula doesn't help much (the only example I know of in our regular calendar is Easter, which is based on the phase of the moon in addition to calendar data; Easter is approximately the first Sunday after the first full moon after the spring equinox, although the full story is even more complicated: see "Ten Divisions Lead to Easter" in *Puzzles and Paradoxes* by T. H. O'Beirne. There are many more examples in the Christian calendar because many special days in the Christian calendar depend on the date of Easter. Holidays from other religions which use a lunar calendar, e.g., Judaism, would work too.)

8. The error is that $(a - c)^2 = (b - c)^2$ does not necessarily imply $a - c = b - c$. You could just as easily (and in fact, more correctly in this case) have $a - c = -(b - c)$. Note how a single small error anywhere in the argument can contaminate the final result, which is why mathematics is so fussy about ensuring that each step is correct.

9. Yes, the square of any odd number is one more than a multiple of 8 is. There are many ways of proving that assertion. Here are two possibilities:

(a) Any odd integer $n$ can be represented in the form $n = 2k + 1$, where $k$ is an integer. Then $(2k+1)^2 = 4k^2 + 4k + 1 = 4k(k+1) + 1$. That certainly proves that the square of an odd integer is one more than a multiple of 4. To go further we can note that $k(k+1)$ is always multiple of 2 (if $k$ is not a multiple of 2, then $k+1$ is).

Alternatively, you could have started with two cases, $n = 4k+1$ and $n = 4k+3$. In the first case, $n^2 = 16k^2 + 8k + 1 = 8k(2k+1) + 1$ which is one more than a multiple of 8; in the second case, $n^2 = 16k^2 + 8k + 9 = 8(2k^2 + k + 1) + 1$ which is also one more than a multiple of 8.

(b) (I like this solution better than the previous solutions.) An odd number must be congruent to one of 1, 3, 5, or 7 mod 8. However, $1^2 = 1 \equiv 1 \bmod 8$, $3^2 = 9 \equiv 1 \bmod 8$, $5^2 = 25 \equiv 1 \bmod 8$, and $7^2 = 49 \equiv 1 \bmod 8$. So the square of any odd number is congruent to 1 mod 8, i.e., is one more than a multiple of 8.

10. It doesn't; let $n = 29$. Then we can pull out a factor of 29: $2n^2 + 29 = 2(29)^2 + 29 = 29 \times (2(29) + 1) = 29 \times 59$ which is not prime.

In general, no polynomial formula can give all primes because of an argument based on the above reasoning. (One has to take a little care to ensure that neither of the factors is 1.) The current record holder for most primes for consecutive values of $n$ from a polynomial is $n^2 + n + 41$ which generates primes for 80 consecutive values of $n$ (if negative values for $n$ are allowed). There are many other related results including the very suprising result that there is a multi-variable polynomial which generates an infinite number of *positive* values all of which are prime! (If it generates a composite number, that number is guaranteed to be negative.)