MATH 221 001 200530 Problem Set 3

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1. **The complement of a set.** If A is a subset of a “universe” set X, then the complement of A in X is defined to be the set of elements of X that are not in A. If X is understood, the complement of A in X is often denoted A’. (See exercises 2.3.3 and 2.4.4 in the textbook.). Find the following, where the universe set X is the set of all integers:
   a. the complement of the set of odd integers
   b. the complement of the set of prime numbers
   c. the complement of the set of composite numbers
   d. the complement of the set of all integers the squares of which are greater than 100.
2. **If and only if.** Define a new operator “if and only if” with the symbol ↔ by the truth table

p	q	p ↔q
F	F	T
F	T	F
T	F	F
T	T	T

   Show that “p ↔ q” is logically equivalent to “(p → q) ^ (q → p)” by finding the truth table for the latter and comparing with the above truth table. (See section 3.4 of the textbook.)
3. Find an unlimited supply of counter-examples to the universal statement “6n-1 is prime for all positive integers n”. (Hint: use modular arithmetic to show that 6n-1 is a multiple of 5 (and larger than 5) for infinitely many values of n.)
4. Write down the contrapositive of the statement *s* = “if n is greater than 3 and n is prime, then n+1 is not prime.” Is the statement *s* true or false? (Supply a proof or counter-example.) Is the contrapositive of *s* true or false?
5. For integers d, m, and n, prove that:
   a. if d divides gcd(m,n) then d divides m and d divides n; and, conversely
   b. if d divides m and d divides n, then d divides gcd(m,n).
6. For any integer m, let D_m be the set of all divisors of the number m. For example, D_{12}={-12,-6, -4, -3, -2, -1, 1, 2, 3, 4, 6, 12}. Prove that for any integers m and n, the intersection of D_m and D_n is $D_{gcd(m,n)}$. (Hint: you can use the result of the previous problem.)
7. Prove that the universal statement “for all sets A, B, and C, A ∩ (B ∪ C) = A ∪ (B ∩ C)” is false by constructing a counter-example.
8. Prove that for all odd integers a, there are integers b and c such that (a,b,c) is a “Pythagorean triple”, i.e., has the property $a^2 + b^2 = c^2$. (Hint: see problem 1.7.3 in the textbook.)
9. On the island of Smulland there live only two types of people, Knights and Knaves. Knights always tell the truth, and Knaves always lie. John and Bill are residents of the island. John says, “If Bill is a Knave then I am a Knight.” Bill says, “We are different”. Is John a Knave or a Knight? Is Bill a Knave or a Knight? (Hint: let J be the statement “John is a Knight” and let B be the statement “Bill is a Knight”, construct a truth table, and decide which lines of the table are self-contradictory.)
10. On the island of Smulland, people are born either Knights or Knaves and remain so for all of their lives. Furthermore, Knights and Knaves remember everything they’ve ever said. I met a resident of the island of Smulland who said, “This is not the first time I have said what I am now saying.” Is that person a Knight or a Knave?