MATH221-001 200530 Problem Set 4 Solutions

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1. This solution is far more detailed than what I expect from you; I am just going into the level of detail to show you what is really necessary if we are to be correct when applying the axiomatic method. Even so, I have skipped some steps like showing that we can drop the brackets in expressions like $a+(b+c)$. (We have to be careful about the meaning of expressions like $4uv$ and $a+b+c$: do we mean $(4u)v$ or $4(uv)$ in the former case, and $(a+b)+c$ or $a+(b+c)$ in the latter? Fortunately, it does not matter because $4(uv) = (4u)v$ by axiom 6 and $a+(b+c) = (a+b)+c$ by axiom 4.) Furthermore, we need to define subtraction in terms of addition (alternatively you could just list a sufficient number of properties of subtraction in new axioms, but that is not necessary and we want to keep the number of axioms to a minimum). We define $a-b = a+(-b)$. I gloss over details like that in the proof below.

   Since we are going to need to calculate squares of two different expressions, we begin with a lemma:

   Lemma. *For all natural numbers x and y, $(x+y)^2 = x^2+2xy+y^2$.*

   Proof. By axiom 9,
   $$(x+y)\times(x+y) = ((x+y)\times x)+((x+y)\times y).$$
   We want to apply axiom 9 again to the (outer) brackets in the above expression, but we can't because they are in the wrong order. So we first apply axiom 5 to switch the order then axiom 9:
   $$((x+y)\times x)+((x+y)\times y) = (x\times(x+y))+(y\times(x+y)) = ((x\times x)+(x\times y))+((y\times x)+(y\times y)).$$
   Now we use axiom 5 to replace yx with xy and axiom 4 to drop the brackets in the addition:
   $$(x^2+(xy))+((yx)+y^2) = (x^2+(xy))+((xy)+y^2) = x^2+(xy)+(xy)+y^2$$
   Finally, we add the two quantities in the middle by axiom 7 and axiom 9:
   $$x^2+(xy)+(xy)+y^2 = x^2+(xy)1+(xy)1+y^2 = x^2+(xy)(1+1)+y^2 = x^2+(xy)2+y^2 = x^2+2xy+y^2$$
   using axiom 5 in the last step. □

   Now by the above lemma, the left hand side of the given expression is
   $$(u-v)^2+4uv = u^2+2u(-v)+(-v)^2+4uv = u^2-2uv+v^2+4uv = u^2-2uv+4uv+v^2$$
   using axiom 3, and then by axioms 5 and 9,
   $$u^2-2uv+4uv+v^2 = u^2+uv(-2)+uv(4)+v^2 = u^2+uv(2)+v^2 = u^2+2uv+v^2,$$
   as required.

2. By the definition of inequality, $b = a+m$ for some natural number m. Multiplying on the right by k, $bk = (a+m)k = ak+mk$, from which it follows by the definition that $ak < bk$.

   Letting $k = a$ in the above and applying axiom 5 we have $a^2 < ba = ab$. Letting $k = b$ in the above we have $ab < b^2$. Then $a^2 < ab < b^2$ and we are done. (Again, I have glossed over some details, in particular that $x < y$ and $y < z$ implies $x < z$; can you fill in the missing steps?)

p	q	r	p	$q \vee r$	$p \wedge (q \vee r)$
F	F	F	F		F
F	F	T	F		F
F	T	F	F		F
F	T	T	F		F
T	F	F	T	F	F
T	F	T	T	T	T
T	T	F	T	T	T
T	T	T	T	T	T

Table 1: Truth table for $p \wedge (q \vee r)$

3. The truth table for $p \wedge (q \vee r)$ appears in Table 1. Note that I have left some entries blank; the truth values of those entries is not important for the result (why not?). The technique is called 'short circuiting', as I mentioned briefly in the lectures. You can fill in the appropriate truth values yourself if it bothers you. Similarly, the truth table for $(p \wedge q) \vee (p \wedge r)$ appears in Table 2. Note that I have duplicated some columns to lessen the chances of making an error, and I have used a different kind of short-circuiting. Since the final rows of the two tables are identical, the two expressions are logically

p	q	r	$p \wedge q$	p	r	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
F	F	F	F	F	F	F	F
F	F	T	F	F	T	F	F
F	T	F	F	F	F	F	F
F	T	T	F	F	T	F	F
T	F	F	F	T	F	F	F
T	F	T	F	T	T	T	T
T	T	F	T	T	F		T
T	T	T	T	T	T		T

Table 2: Truth table for $(p \wedge q) \vee (q \wedge r)$

equivalent.

You can lessen the amount of work you have to do by combining this all in one grand table; I have chosen not to because the method is clearer when presented in two tables.

4. By solution 3,

$$\begin{aligned} x \in (A \cap (B \cup C)) &\Leftrightarrow (x \in A) \wedge ((x \in B) \vee (x \in C)) \\ &\Leftrightarrow ((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C)) \\ &\Leftrightarrow x \in ((A \cap B) \cup (A \cap C)). \end{aligned}$$

5. The two expressions have equal columns in Table 3 so they are logically equivalent.

6. By solution 5,

$$\begin{aligned} x \in (A \cup B)' &\Leftrightarrow \neg(x \in A \vee x \in B) \\ &\Leftrightarrow \neg(x \in A) \wedge \neg(x \in B) \\ &\Leftrightarrow x \in (A' \cap B'). \end{aligned}$$

7. Let $P(n)$ be the statement $\displaystyle\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
F	F	F	T	T	T	T
F	T	T	F	T	F	F
T	F	T	F	F		F
T	T	T	F	F		F

Table 3: Truth table for $\neg(p \vee q)$ and $\neg p \wedge \neg q$

Base case. $P(1)$ means $1(1+1) = \dfrac{1(1+1)(1+2)}{3}$ which is true because both sides are 2.

Induction hypothesis. Suppose $P(k)$ holds for some k, i.e., $\sum_{i=1}^{k} i(i+1) = \dfrac{k(k+1)(k+2)}{3}$ for some k.

Induction step. Under the above assumption we need to prove $P(k+1)$. The left hand side of $P(k+1)$ is

$$\sum_{i=1}^{k+1} i(i+1) = \sum_{i=1}^{k} i(i+1) + (k+1)((k+1)+1).$$

By the induction hypothesis, the above expression is equal to

$$\frac{k(k+1)(k+2)}{3} + (k+1)(k+2) = (k+1)(k+2)\left(\frac{k}{3}+1\right) = (k+1)(k+2)\frac{k+3}{3} = \frac{(k+1)(k+2)(k+3)}{3}.$$

On the other hand, the right hand side of $P(k+1)$ is

$$\frac{(k+1)((k+1)+1)((k+1)+2)}{3} = \frac{(k+1)(k+2)(k+3)}{3}.$$

Since the left hand and right hand sides of $P(k+1)$ agree, $P(k+1)$ is true under the assumption that $P(k)$ is true, showing that $P(k) \Rightarrow P(k+1)$.

Therefore by induction, $P(n)$ is true for all $n \in \mathbb{N}$.

8. Let $Q(n)$ be the statement $\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$.

   Base case. The left hand side of $Q(1)$ is $f_1^2 = 1$ and the right hand side is $f_1 f_2 = 1 \cdot 1 = 1$. Since the two sides agree, $Q(1)$ is true.

   Induction hypothesis. Suppose that $Q(k)$ is true for some k, i.e., that $\sum_{i=1}^{k} f_k^2 = f_k f_{k+1}$ is true for some natural number k.

   Induction step. Under the above assumption we need to prove $Q(k+1)$. The left hand side of $Q(k+1)$ is

$$\sum_{i=1}^{k+1} f_k^2 = \sum_{i=1}^{k} f_k^2 + f_{k+1}^2$$

   By the induction hypothesis, the above expression is equal to

$$f_k f_{k+1} + f_{k+1}^2 = f_{k+1}(f_k + f_{k+1}) = f_{k+1} f_{k+2}$$

   where we have used the recursive definition of the Fibonacci numbers in the last step.

   On the other hand, the right hand side of $Q(k+1)$ is $f_{k+1} f_{(k+1)+1} = f_{k+1} f_{k+2}$. Since the two sides agree, $Q(k+1)$ is true under the assumption that $Q(k)$ is true, establishing the induction step $Q(k) \Rightarrow Q(k+1)$.

   Therefore by induction $Q(n)$ is true for all $n \in \mathbb{N}$.

9. Let's try a few to see if we can find a pattern.

$$\begin{aligned} n=1: &\quad 1 = 1 \\ n=2: &\quad 4 = 1+3 \\ n=3: &\quad 9 = 1+3+5 \\ n=4: &\quad 16 = 1+3+5+7 \\ n=5: &\quad 25 = 1+3+5+7+9. \end{aligned}$$

It would appear that $\sum_{i=1}^{n}(2i-1) = n^2$. (Aside: there's that minus sign sneaking in there again. Question: can you restructure the question so that subtraction isn't required anywhere?) Let's prove our guess by induction. Below I have given an abbreviated argument which you can recast in the more formal format used above if you find that helpful.

We have already checked the base case (and several beyond the base case). Assuming that $\sum_{i=1}^{k}(2i-1) = k^2$, let's prove the result for the next value of k: $\sum_{i=1}^{k+1}(2i-1) = \sum_{i=1}^{k}(2i-1) + (2(k+1)-1)$ which by the induction hypothesis is equal to $k^2 + (2(k+1)-1) = k^2 + 2k + 1 = (k+1)^2$, so the result is true for all $n \in \mathbb{N}$ by induction.

The result is illustrated in Figure 1, which is quite convincing even though it wouldn't be acceptable as a formal proof of the result.

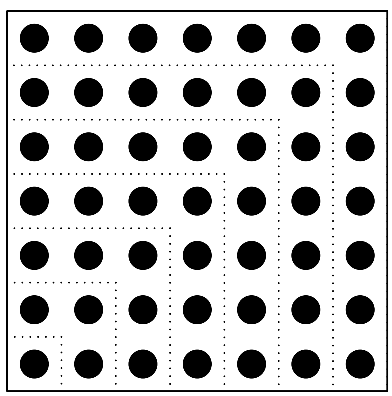

Figure 1: "Proof without words" that $1 + 3 + 5 + \cdots + (2n-1) = n^2$

10. If we are allowed to use negative stamps, then we could make any multiple of $\gcd(5, 12) = 1$, i.e., any value whatsoever. The restriction that we can only use nonnegative numbers of stamps means that certain values can't be reached, such as 18 cents, as pointed out in the problem statement. Let's investigate the situation by assembling a table of the possible values of $5m + 12n$ for $m, n = 0, 1, 2, \ldots$. It's easy to build up the table across the rows: every element is larger by 10 than the element two spaces to the left.

Making a list of all the values we've got so far, we obtain the set

$$S = \{0, 5, 10, 12, 15, 17, 20, 22, 24, 25, 27, 29, 30, 32, 34, 36, 37, 39, 40, 41, 42, 44, 45, 46, 47, 49, \ldots\}.$$

A lot of the low values are missing, but it looks like once we get into the 40's we see that most of the values are there. Only 43 and 48 are missing. In fact, we don't see any numbers ending in 3 or 8, but

		m									
		0	1	2	3	4	5	6	7	8	9
n	0	0	5	10	15	20	25	30	35	40	45
	1	12	17	22	27	32	37	42	47	52	57
	2	24	29	34	39	44	49	54	59	64	69
	3	36	41	46	51	56	61	66	71	76	81

Table 4: Table of values of $5m + 12n$, $m = 0, \ldots, 9$, $n = 0, \ldots, 3$

if we add another row to the table we see 48, 53, and every number ending in 3 or 8 thereafter. This gives us a way of organizing the information we have so far: it seems we have every number ending in 0 or 5, every number ending in 2 or 7 except for 2 and 7 themselves, every number ending in 4 or 9 except for 4, 9, 14, and 19, every number ending in 6 or 1 except for 1, 6, 11, 16, 21, 26, and 31, and every number ending in 3 or 8 except for 3, 8, 13, 18, 23, 28, 33, 38, and 43.

It seems clear that there's no way we can get the missing numbers by extending the table, so we guess that the entire list of missing numbers is

$$M = \{2, 7, 4, 9, 14, 19, 1, 6, 11, 16, 21, 26, 31, 3, 8, 13, 18, 23, 28, 33, 38, 43\}.$$

Being skeptical, we should actually try to prove that M is the entire list of missing numbers. This is where we use strong induction.

Theorem. *Every number $s \geq 44$ can be written in the form $s = 5m + 12n$ for some nonnegative integers m and n.*

Proof. We argue by strong induction. We need to check the first five cases for the base: $44 = 5 \times 4 + 12 \times 2$, $45 = 5 \times 9 + 12 \times 0$, $46 = 5 \times 2 + 12 \times 3$, $47 = 5 \times 5 + 12 \times 1$, and $48 = 5 \times 0 + 12 \times 4$. Let the induction hypothesis be, "all numbers from 44 to k can be represented in the form $5m + 12n$". Then for $k \geq 48$ it follows that $k - 4$ can be represented in the form $5m + 12n$, so $k + 1 = k - 4 + 5 = 5m + 12n + 5 = 5(m + 1) + 12n$ can also be represented in the form $5m' + 12n'$, establishing the induction step. □

Note that at any step we go five steps back, so we need five initial steps to get us started.

Question: can we do this for numbers p and q other than 5 and 12? What about three numbers p, q, and r?