MATH221-001 200530 Problem Set 4

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1. Construct a proof of the following rule using only axioms 1–9 of chapter 4, stating clearly at each step which axiom you are using:
$$(u+v)^2 = (u-v)^2 + 4uv. \tag{1}$$
(Recall that a^2 just means $a \times a$.)

2. Using only axioms 1–10 of chapter 4, prove that for all natural numbers a and b, $a < b$ implies $a^2 < b^2$. (Hint: first prove that $a < b$ implies $ak < bk$ for any natural number k. Let $k = a$, etc.)

3. Use truth tables to show that the logical expressions $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ are logically equivalent. (Hint: You'll have to construct one or two truth tables with 8 rows for all the $2^3 = 8$ combinations of p, q, and r being True or False. Counting in binary from 0 to 7: $(000)_2$, $(001)_2$, $(010)_2$, $(011)_2$, $(100)_2$, $(101)_2$, $(110)_2$, and $(111)_2$.)

4. **The correspondence between logic and set theory.** Consider the set $A \cap (B \cup C)$. By the definitions of intersection and union, $x \in A \cap (B \cup C)$ means $x \in A \wedge (x \in B \vee x \in C)$, so there is a direct correspondence between the set theoretical expression and the logical expression. Use question 3 to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. (Hint: show that if $x \in A \cap (B \cup C)$, then $x \in A \wedge (x \in B \vee x \in C)$, then $(x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$, then $x \in (A \cap B) \cup (A \cap C)$; also show the converse.)

5. Use truth tables to show that the logical expressions $\backsim (p \vee q)$ and $\backsim p \wedge \backsim q$ are logically equivalent.

6. **The correspondence between 'not' and complement.** Suppose the sets A and B are subsets of some 'universe' set X, and that we define the complement of A, A', to be the set of all elements of X which are not in A. We define B' similarly. Then there is a direct correspondence between the statements $x \in A'$ and $\backsim (x \in A)$. Use question 5 to show that $(A \cup B)' = A' \cap B'$ for any subsets A and B of a universe set X.

7. Prove by mathematical induction that
$$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}.$$

8. Let $f_1, f_2, f_3, \ldots$ be the Fibonacci numbers. Prove by mathematical induction that
$$\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}.$$

9. Find a formula for the sum of the odd numbers from 1 to $2n-1$ and prove that your formula is correct by induction.

10. Suppose you have an unlimited amount of 5 and 12 cent stamps. It is possible to make 22 cents in postage with a 12 cent stamp and two 5 cent stamps, but it is impossible to make 18 cents in postage. For what natural numbers k is it possible to make k cents in postage? Prove your claim using mathematical induction.