MATH221-001 200530 Problem Set 5 Solutions DRAFT

Edward Doolittle

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1. (a) The expressions $p \Rightarrow q$ and $q \vee \neg p$ are logically equivalent because the corresponding columns in Table 1 are identical.

p	q	$p \Rightarrow q$	q	$\neg p$	$q \vee \neg p$
F	F	T	F	T	T
F	T	T	T		T
T	F	F	F	F	F
T	T	T	T		T

Table 1: Truth table for $p \Rightarrow q$ and $q \vee \neg p$

(b) By all the logical equivalency rules we have assembled,

$$\neg(p \Rightarrow q) \equiv \neg(q \vee \neg p) \equiv \neg q \wedge \neg(\neg p) \equiv \neg q \wedge p.$$

(c) In the above equivalence, p is "it rains", q is "it pours", so "It is not the case that if it rains, it pours" is equivalent to "It doesn't pour, and it rains" or more idiomatically, "It isn't pouring, but it's raining".

2. (a) The set in question is $\{7, 8, 9, \ldots\}$ so its least member is 7. To construct a formal proof that the least member is 7, you could check that $1, 2, \ldots, 6$ are not in the set but 7 is, but we'll let it go without a formal proof.

(b) The set in question is $\{40, 41, 42, \ldots\}$ so its least member is 40. Again it's OK if you just list the elements of the set, but I'm starting to get less comfortable with implicit justification. My personal judgment is that the situation is still simple enough that an explicit argument isn't necessary, but it's starting to get less clear. Below is the explicit argument I would use if pressed.

If $n \geq 40$ then by the rules for inequalities we worked out earlier, we can multiply the inequality by n to obtain $n^2 \geq 40n$. Therefore the set in question contains every number from 40 up. We should check that any number less than 40 is out of the set; we can do that in the same way. If $n < 40$ then by the rules for inequalities we have learned $n^2 < 40n$ for any natural number, so every natural number less than 40 is out of the set. Therefore the least member of the set is 40.

(c) The set in question is $\{4, 5, 6, \ldots\}$, so the least member is 4, but now I'm getting really uncomfortable with the implicit argument. I suggest you provide a proper argument for this question, but we'll still accept a list as a complete answer.

For a proper argument, you could use induction exactly as in question 6, or you could use integers instead of natural numbers and rewrite the inequality $(n-1)(n-3) > 0$ which is true if and only if $(n > 1$ and $n > 3)$ or $(n < 1$ and $n < 3)$, i.e., if and only if n is in the set $\{4, 5, 6, \ldots\} \cup \{0, -1, -2, \ldots\}$. The intersection of the natural numbers with the solution set is $\{4, 5, 6, \ldots\}$ which is the set in question; its least member is clearly 4.

(d) The set of numbers is given by the table in solution 10 of problem set 4, minus the 0 row and 0 column. The smallest number is 17.

Again, I'm not totally comfortable with the implicit argument. I will accept just a list as a correct answer, but I would be happier if you argued as follows: since x and y are natural numbers, $x \geq 1$ and $y \geq 1$, which implies that $5x \geq 5$ and $12y \geq 12$, so $5x + 12y \geq 5 + 12 = 17$. That shows that the least element of the set is greater than or equal to 17; since 17 is in the set, 17 must be the least element of the set.

3. Base: when $n = 1$ the left side is $1 + 2^1 = 3$, the right side is $2^{1+1} - 1 = 4 - 1 = 3$, and the two sides are equal, proving the statement when $n = 1$.

   Induction step: assume $1 + 2^1 + \cdots + 2^k = 2^{k+1} - 1$ for some k. Then $1 + 2^1 + \cdots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1} = 2 \times 2^{k+1} - 1 = 2^{(k+1)+1} - 1$ which proves the induction step. The result follows for all $n \in \mathbb{N}$ by induction.

4. Base: when $n = 1$, $3^{3n} - 1 = 3^3 - 1 = 26$ is a multiple of 13 so the statement is true for the base case.

   Induction step: if $3^{3k} - 1$ is a multiple of 13 we can write $3^{3k} = 13m + 1$ for some m. Then $3^{3(k+1)} = 3^{3k} \times 3^3 = (13m + 1) \times 27$. It follows that $3^{3(k+1)} - 1 = 13 \times 27m + 26 = 13(27m + 2)$ is a multiple of 13, proving the induction step. The result is true for all $n \in \mathbb{N}$ by induction.

5. By the theorems of modular arithmetic, $3^3 \equiv 27 \equiv 1 \bmod 13$ so $3^{3n} \equiv (3^3)^n \equiv 27^n \equiv 1^n \equiv 1 \bmod 13$. Therefore for all $n \in \mathbb{N}$, $3^{3n} - 1 \equiv 0 \bmod 13$, i.e., $3^{3n} - 1$ is a multiple of 13 for all $n \in \mathbb{N}$.

6. Base: when $n = 4$, the left side of the inequality is 16 and the right side is $3 \times 4 + 2 = 14$, so the inequality is true for $n = 4$.

   Induction step: suppose $k^2 > 3k + 2$. Then the left side of the next inequality is $(k+1)^2 = k^2 + 2k + 1$ and the right side is $3(k+1) + 2 = 3k + 2 + 3$. We need to prove that the left side is greater than the right side. By the induction hypothesis, $k^2 > 3k + 2$, so if we can show $2k + 1 > 3$, i.e., $2k > 2$, i.e., $k > 1$, we're done. But since we're starting our induction at $n = 4$, we know $k \geq 4 > 1$ which establishes the required inequality.

   Strictly speaking, we have provided the analysis for the induction step, not the proof. Since the proof is implicit from the analysis, that is enough, but for the sake of completeness I will provide the proof by working the analysis backwards. Suppose the induction hypothesis, i.e., $k^2 > 3k + 2$ for some natural number $k \geq 4$. Then we have $k \geq 4 > 1$ which implies $2k > 2$ which implies $2k + 1 > 3$. Adding that inequality to the induction hypothesis we obtain $k^2 + 2k + 1 > 3k + 2 + 3$, i.e., $(k+1)^2 > 3(k+1) + 2$, which proves the induction step.

   The difficulty with the proof alone is that it seems to come out of thin air: how did I know to use $k > 1$ instead of $k > 4$? The difficulty with analysis alone is that it isn't what's required from the logical perspective. Given a choice between the two, a mathematician will usually provide the analysis because the proof can be constructed from the analysis in a straightforward way, but one should keep in mind that for total correctness an explicit proof must be supplied.

7. Testing a few, $1! = 1 < 3^1 = 3$, $2! = 2 < 3^2 = 9$, $3! = 6 < 3^3 = 27$, $4! = 24 < 3^4 = 81$, $5! = 120 < 3^5 = 243$, $6! = 720 < 3^6 = 729$, but $7! = 5040 > 3^7 = 2187$ so $n_0 = 7$. That establishes the base case for the induction argument.

   Induction step: suppose $k! > 3^k$ for some $k \geq 7$. Then we have $k+1 > 7 > 3$. Multiplying the induction hypothesis inequality by the inequality $k + 1 > 3$ we obtain $(k+1)k! > 3 \times 3^k$, i.e., $(k+1)! > 3^{k+1}$ which establishes the induction step.

8. We need to prove that $1 \leq n$ for all $n \in \mathbb{N}$. Base: $1 \leq 1$ is clearly true. Induction step: suppose $1 \leq k$ for some $k \in \mathbb{N}$. Then $k < k + 1$ because there is a natural number m such that $k + m = k + 1$ (namely $m = 1$). Then $1 \leq k < k + 1$ shows that $1 \leq k + 1$, establishing the induction step and showing the result is true for all $n \in \mathbb{N}$.

9. To establish the next case we're going to need something like the three previous cases, so let's first establish a base of the first three cases. $a_1 = 1 = 2^{1-1}$, so $a_n \leq 2^{n-1}$ holds for $n = 1$. Second

$a_2 = 2 = 2^{2-1}$, so $a_n \leq 2^{n-1}$ is true for $n = 2$ as well. Third, $a_3 = 3 < 2^{3-1} = 4$, so $a_n \leq 2^{3-1}$ is true for $n = 3$.

Now to establish the induction step we'll find the result of problem 3 helpful. Assume that the result is true for $n = 1, 2, \ldots, k, k+1, k+2$. We need to show that it is true for $n = k+3$. By the strong induction hypothesis applied to k, $k+1$, and $k+2$,

$$a_{k+3} = a_{k+2} + a_{k+1} + a_k \leq 2^{k+1} + 2^k + 2^{k-1} \leq 2^{k+1} + 2^k + 2^{k-1} + \cdots + 2^1 + 2^0 = 2^{k+2} - 1 \leq 2^{(k+3)-1}$$

which shows that the result is true for $n = k+3$. By strong induction, the result is true for all $n \in \mathbb{N}$.

10. The algebra is straightforward: $x^2 = 2y^2$ implies that

$$2z^2 = 2(x-y)^2 = 2x^2 - 4xy + 2y^2 = x^2 - 4xy + x^2 + 2y^2 = x^2 - 4xy + 4y^2 = (x-2y)^2 = w^2.$$

We also need to check that $z = x - y$ and $w = 2y - x$ are natural numbers. Clearly they are integers since x and y are integers, so we just need to check whether $z > 0$ and $w > 0$. We argue by contradiction (i.e., prove the contrapositive statement). If $x \leq y$ then $x^2 \leq y^2 < 2y^2$ which contradicts our assumption that $x^2 = 2y^2$, so we must have $x > y$, so $z > 0$. Similarly, if $x \geq 2y$ then multiplying through by $x > y$ (which we know is true from the previous argument) we have $x^2 > 2y^2$. That again condradicts our assumption that $x^2 = 2y^2$, so again we must have $x < 2y$, i.e., $w > 0$.

In order to show that S has no least element, we argue as follows. Suppose $x \in S$. Then there is a natural number y such that $x^2 = 2y^2$. By the above, we can construct another pair of natural numbers (w, z) with the property that $w^2 = 2z^2$. That means that we also have $w \in S$. We can repeat the above reasoning to construct another pair $(w_1, z_1) = (2z - w, w - z)$ with the property that $w_1^2 = 2z_1^2$ and so on for (w_2, z_2), (w_3, z_3), ... Furthermore, we have $w = 2y - x < x$ because $2y < 2x$ from the above; by the same kind of reasoning $w > w_1 > w_2 > \cdots$ which shows that our set S has no least member. By theorem 4.7 in the textbook, S must be empty; there is no integer x such that $x^2 = 2y^2$ for some integer y.