MATH221-001 200530 Problem Set 5

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Due: Friday, November 4, 2005

1. (a) Show that $p \Rightarrow q$ is logically equivalent to $q \vee \neg p$.

   (b) Use the above result and the logical identities $\neg(\neg r) \equiv r$ and $\neg(s \vee t) \equiv (\neg s) \wedge (\neg t)$ to find an expression using only the logical operators $\neg$ and $\wedge$ which is logically equivalent to $\neg(p \Rightarrow q)$.

   (c) Use the above result to write out the negation in words of the statement "If it rains, it pours" without using the words 'if', 'then', or 'implies'.

2. For each of the following sets of natural numbers, find its least member.

   (a) $\{n \in \mathbb{N} : n^2 \geq 38\}$

   (b) $\{n \in \mathbb{N} : n^2 \geq 40n\}$

   (c) $\{n \in \mathbb{N} : n^2 + 3 > 4n\}$

   (d) $\{n \in \mathbb{N} : n = 5x + 12y \text{ for some } x, y \in \mathbb{N}\}$

3. Use weak induction to prove that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for all $n \in \mathbb{N}$.

4. Prove that $3^{3n} - 1$ is a multiple of 13 for all $n \in \mathbb{N}$ by induction. (Hint: show that 3^{3n} is of the form $13m + 1$.)

5. Prove that $3^{3n} - 1$ is a multiple of 13 for all $n \in \mathbb{N}$ using modular arithmetic. (Hint: show that $3^{3n} \equiv 1 \bmod 13$.

6. Show that $n^2 > 3n + 2$ for all integers $n \geq 4$ by induction. (Hint: see the first example on page 33 of the textbook.)

7. Find the least natural number n_0 for which $n! > 3^n$. Prove that the inequality is true for all $n \geq n_0$ by induction.

8. Prove by induction that 1 is the least member of $\mathbb{N}$.

9. Let the numbers a_n be defined recursively by $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and

$$a_{n+3} = a_{n+2} + a_{n+1} + a_n$$

   for $n = 1, 2, 3, \ldots$. Use strong induction to prove that $a_n \leq 2^{n-1}$ for $n \in \mathbb{N}$.

10. **The method of infinite descent.** Show that if there are natural numbers x and y such that $x^2 = 2y^2$, then it is also true that $w^2 = 2z^2$ where $w = 2y - x$ and $z = x - y$ are natural numbers.

    Now let $S = \{x \in \mathbb{N} : x^2 = 2y^2 \text{ for some } y \in \mathbb{N}\}$. Prove that S has no least element and therefore must be empty. (This shows that there are no natural numbers x and y that satisfy $x^2 = 2y^2$; the argument may be related to the geometrical reasoning that the Pythagoreans originally used to prove that $\sqrt{2}$ is irrational.)