MATH221-001 200530 Problem Set 6 Solutions DRAFT

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1. Genetically speaking, f must be a function, since a person can have at most one mother (and I'm willing to wager that everyone in the class has a mother in the genetic sense, although if one believes any of a number of different creation stories there are some people without mothers).

   Legally speaking, g must be a function because a person can have no more than one (legal) husband. I accept g as a function, but the textbook doesn't because there are some elements of S that don't have husbands (all the men and many of the women, I imagine).

   Whether h is a function or not depends on whether there is a student in the class who has more than one child. A bonus point for anyone who did the necessary research.

   The only relation of the four which is definitely not a function is k because every one of you I'm sure has two (genetic) grandmothers.

2. To really drive this home you need a specific counter-example. Pick any number at random, say $n = 1$. Then $s(t(n)) = s(4) = 16$ but $t(s(n)) = t(1) = 4$. Since $s \circ t(1) \neq t \circ s(1)$ the two functions are not equal at 1 so the are not equal.

3. (a) $T(1) = 3(1)+1 = 4$, $T(2) = 2/2 = 1$, $T(3) = 3(3)+1 = 10$, $T(4) = 4/2 = 2$, $T(5) = 3(5)+1 = 16$, $T(6) = 6/2 = 3$.

   (b) T is not an injection because $T(1) = 4 = T(8)$, for example. T is a surjection because $T(2n) = n$ for any $n \in \mathbb{N}$. T is not a bijection because it is not an injection.

   (c) T does not have a left inverse because it is not an injection. (You could also argue that if T did have a left inverse g, we would have to have both $g(4) = 1$ and $g(4) = 8$ which is impossible, so there is no such function g.)

   Since T is a surjection it does have a right inverse h. (In fact, we already have one explicit example of such a right inverse: $h(n) = 2n$. Are there others?)

   Since T is not an injection (and does not have a left inverse), it does not have an inverse.

4. The first four thieves have the effect $f(n) = (n/2) - 1$ on the number of rubies; in order for the function to make sense, it must be a mapping $f : 2\mathbb{N} \to \mathbb{N} \cup \{0\}$ from the even natural numbers to the nonnegative integers. (Alternatively, as I mentioned, you could consider f as a mapping from $\mathbb{Q}$ to $\mathbb{Q}$ or $\mathbb{R}$ to $\mathbb{R}$, but then we'll have to worry about the range of f^{-1} later.) It is easy to show that f is an injection and a surjection, so it is a bijection, so it has an inverse, the formula of which must be $f^{-1}(m) = 2(m+1)$. The number of rubies before the first thief entered must be

$$f^{-1}(f^{-1}(f^{-1}(f^{-1}(1)))) = f^{-1}(f^{-1}(f^{-1}(4))) = f^{-1}(f^{-1}(10)) = f^{-1}(22) = 46.$$

   You should check that the number 46 actually satisfies the conditions of the problem.

5. We need to know the number of possible different days of the month on which a person can be born. There is no month with 32 days or more, but there is a month with 31 days, so the number of pigeonholes is 31. If we had 62 people in the room, we could fit two into each pigeonhole and not have three people born on the same day of the month, so 62 is smaller than the number we want. On the other hand if we have 63 people in the room, there must be three born on the same day of the month.

(I don't require a proof of the above fact, but it is not a completely straightforward application of the pigeonhole principle we have learned. See if you can prove this result using induction or the pigeonhole principle that we have already proven.)

6. There are actually 366 different possible birthdays (a person could have their birthday on February 29), so the pigeonhole principle tells us that we need 367 people in the room to prove that there are two with the same birthday. (A question for those of you who have studied probability and statistics: how many people do we need in the room to ensure that it is more than 50

7. Let f be the mapping from subset of $\mathbb{N}_{10}$ to $\mathbb{N}_{10}$ given by $f(S) = S\setminus\{10\}$, i.e., if 10 is in S we take it out, otherwise we leave S alone. The range of f is the collection of all subsets of $\mathbb{N}_9$ which has $2^9 = 512$ elements, so by the pigeonhole principle, two elements of a collection of 513 subsets must have the same target. That is only possible if the two sets are the same except that one contains 10 and the other doesn't; it follows that one set is a subset of the other. (Question: do you think we can draw the same conclusion with fewer than 513 subsets? Try to construct 512 subsets of $\mathbb{N}_{10}$ none of which is a subset of another. First try the same for subset of $\mathbb{N}_n$ where n is smaller than 10. This is a fairly difficult problem.)

8. Let $g = f \circ f$, i.e., $g : X \to X$ is defined by $g(x) = f(f(x))$. Then $f(g(x)) = f(f(f(x))) = x$ and $g(f(x)) = f(f(f(x))) = x$ for all $x \in X$, so f has a (two-sided) inverse. It follows that f must be a bijection. (In detail, f is injective because $f(x) = f(y)$ implies $g(f(x)) = g(f(y))$ which implies $x = y$, and f is surjective because $y \in X$ implies $y = f(x)$ where $x = g(y)$.)

9. A covering with dominos gives a bijection between the black squares and white squares of a chessboard. If we remove two diagonally opposite squares, we are removing two squares of the same colour, so a covering in that case would be a bijection between 30 squares of one colour and 32 squares of the other colour, which is impossible by the pigeonhole principle. Therefore there can be no such covering.

10. Since $f(n) < n$ for all $n \in \mathbb{N}$ we can form a new function $f_{10} : \mathbb{N}_{10} \to \mathbb{N}_9$, defined by $f_{10}(x) = f(x)$ for any $x \in \mathbb{N}_{10}$. (f_{10} is called the *restriction* of f to $\mathbb{N}_{10}$.) Therefore f cannot be an injection by the pigeonhole principle. (The number 10 above was arbitrary. Furthermore, the more the function compresses, the more radically it fails injectivity (as we'll see when we learn to count functions). No scheme can compress functions losslessly, and in fact PKZip and other lossless compression schemes actually *increase* the size of files on average; they are designed so that common file formats like plain text are compressed while random-looking files of little interest to humans are the files that are expanded under the scheme.)