MATH221-001 200530 Problem Set 6

Edward Doolittle

Due: Monday, November 14, 2005

1. Let S be the set of students in MATH 221 001 200530 (i.e., this class) and let P be the set of all people who have ever lived on Earth. Which of the following specify functions from S to P?

   (a) $f(x)$ is the mother of x

   (b) $g(x)$ is the husband of x

   (c) $h(x)$ is the child of x

   (d) $k(x)$ is the grandmother of x

2. Let $t : \mathbb{N} \to \mathbb{N}$ be given by $t(n) = 3n+1$, and $s : \mathbb{N} \to \mathbb{N}$ be given by $s(n) = n^2$. Show that $s \circ t \neq t \circ s$.

3. Let $T : \mathbb{N} \to \mathbb{N}$ be defined by the formula $T(n) = \begin{cases} 3n+1 & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even} \end{cases}$.

   (a) Find $T(1), T(2), \ldots, T(6)$.

   (b) Is T an injection? A surjection? A bijection?

   (c) Does T have a left inverse? A right inverse? An inverse?

4. A thief enters into a jewelry shop and takes half the rubies, plus one for good measure. She leaves, and a second thief then enters and takes half the remaining rubies, plus one for good measure. She leaves, and a third and fourth thief each do the same. But when a fifth thief enters the shop, he finds only one ruby (which he takes). Write a function f which represents the effect any of the first four thieves has on the number of rubies in the shop. Find f^{-1} and use it to find the number of rubies in the shop before the first thief entered.

5. How many people do we have to have in a room to ensure that there are three people in the room who were born on the same day of the month? (Someone born on September 6, 1984 'is born on the same day of the month' as someone born on October 6, 1985.)

6. How many people do we have to have in a room to ensure that there are two people in the room with the same birthday? (Someone born on September 6, 1984 'has the same birthday' as someone born on September 6, 1985.)

7. The set $\mathbb{N}_{10}$ has $2^{10} = 1024$ distinct subsets. Suppose we have a collection of 513 of those subsets. Show that there must be two sets A and B in our collection such that $A \subset B$. (Hint: some subsets of $\mathbb{N}_{10}$ don't contain 10 and some subsets do.)

8. Let X be any set. Prove that any function $f : X \to X$ with the property $f(f(f(x))) = x$ for all $x \in X$ must be a bijection.

9. Consider an 8×8 chessboard with squares coloured alternately black and white. Is it possible to cover the chessboard with 32 dominoes, each one covering two squares exactly? Now remove two diagonally opposite corners of the chessboard. Is it possible to cover the resulting board with 31 dominoes?

10. We have a computer program that compresses files. Each computer file can be represented by a number (i.e., the numeric value of the sequence of 0's and 1's in the file, considered as a binary number). Since the program compresses files, it corresponds to a function $f : \mathbb{N} \to \mathbb{N}$ with the property $f(n) < n$ for all $n \in \mathbb{N}$. Show that f cannot be an injection. (We call such compression schemes 'lossy'; e.g., compression of audio to MP3 format is lossy, because some information is lost in the process. What about schemes like 'PKzip' which purport to be lossless?)