MATH221-001 200530 Problem Set 7 Solutions DRAFT

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1. See Table 1. The relationship of brotherhood is not transitive, because if a person is male, his brother's brother could be himself; similarly, a person's sibling's sibling is not necessarily a sibling.

xRy	Reflexive?	Symmetric?	Transitive?	Equivalence Relation?
x is y's brother	N	N	N	N
x is y's sibling	N	Y	N	N
x is y's older sibling	N	N	Y	N
x and y have same last name	Y	Y	Y	Y

Table 1: Properties of various human relations

2. (a) Map the odd natural numbers to natural numbers which are one more than a multiple of 7, and map the even natural numbers to natural numbers which are 3 more than a multiple of 7:

$$f(n) = \begin{cases} 7(n-1)/2 + 1 & \text{if } n \text{ odd} \\ 7n/2 + 3 & \text{if } n \text{ even.} \end{cases}$$

   Then f is an injection because $f(m) = f(n)$ implies $f(m) \equiv f(n) \bmod 7$ which implies that m and n are both odd or both even. In the former case we have $7(n-1)/2 + 1 = 7(m-1)/2 + 1$ which implies $m = n$; similarly in the latter case. In summary, in all cases $f(m) = f(n)$ implies $m = n$.

   Furthermore, f is a surjection onto the set of natural numbers which are congruent to 1 mod 7: if $y \equiv 1 \bmod 7$ then $2(y-1)/7 + 1$ is an odd integer and $f(2(y-1)/7 + 1) = y$. Similarly, if $y \equiv 3 \bmod 7$ then $2(y-3)/7$ is an even integer and $f(2(y-3)/7) = y$. (Here we have shown that f is a surjection by constructing a right inverse.)

   A more informal argument is acceptable, but you should know how to construct a formal argument.

   (b) Pairs of integers can be represented graphically as 'lattice points' in the plane, i.e., points with integer coordinates such as $(2, -3)$. The easiest way I can think of to count the lattice points in order is by following the spiral pattern starting at the origin illustrated in Figure 1. There is an explicit formula for the function mapping $\mathbb{N}$ to $\mathbb{Z} \times \mathbb{Z}$ suggested by the diagram but it is complicated, and I find the diagram convincing enough.

3. (a) We argue by contradiction. Assume that a is rational. Since the difference of two rational numbers is rational, $1 - a = 1 - (1 - \sqrt{2}) = \sqrt{2}$ would be rational, which contradicts what we know about $\sqrt{2}$. Therefore a must be irrational. Similarly for b.

   (b) $a + b = 2$ is rational. $ab = (1 - \sqrt{2})(1 + \sqrt{2}) = 1 - 2 = -1$ is rational.

   (c) $a - b = -2\sqrt{2}$. Assume that $-2\sqrt{2}$ is rational. Then dividing through by the rational number -2 we would have $\sqrt{2}$ rational, contradiction. Therefore $a - b$ is irrational.

4. Let $c = -b$, $d = 1/b$. Then from the above, a and b are irrational; furthermore, c and d are irrational (otherwise b would be rational), and by the above we have $a + b$ rational, $a + c$ irrational, ab rational, and ad irrational.

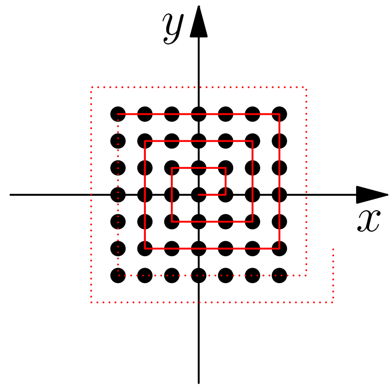

Figure 1: Pathway through $\mathbb{Z} \times \mathbb{Z}$

5. The function is an injection because

$$f(x) = f(y) \implies 1 - \frac{1}{x} = 1 - \frac{1}{y} \implies \frac{1}{x} = \frac{1}{y} \implies x = y.$$

The function is not a surjection because there is no x such that $f(x) = 1$. To prove that fact we can argue by contradiction. Assume $f(x) = 1$ for some $x \in \mathbb{Q}$. Then

$$1 - \frac{1}{x} = 1 \implies \frac{1}{x} = 0 \implies 1 = 0,$$

a contradiction. Therefore there is no $x \in \mathbb{Q}$ such that $f(x) = 1$.

We can calculate an inverse for f by solving the equation $f(x) = y$ for y:

$$1 - \frac{1}{x} = y \implies 1 - y = \frac{1}{x} \implies x = \frac{1}{1-y}$$

so the right inverse of f is $g : \mathbb{Q}\backslash\{1\} \to \mathbb{Q}\backslash\{0\}$ given by $g(y) = 1/(1-y)$ (check). If the point $x = 0$ is excluded from the domain of f, and $y = 1$ is excluded from the range of f, then g is a two-sided inverse for f.

6. If g is a surjection then it has a right inverse $h : X \to X$; i.e., $g(h(x)) = x$ for all $x \in X$. It follows from the cancellation equation that h must be an injection (try the proof yourself), so by the pigeonhole principle it must be a bijection. It follows that h must have a two-sided inverse, call it k, which is necessarily a bijection. Then letting $x = k(z)$ in the cancellation equation, $g(h(k(z))) = k(z)$ for all $z \in X$. Since k is an inverse for h, $g(z) = k(z)$ for all $z \in X$, i.e., $g = k$ is a bijection, as required.

There may be many variations on the above proof, but the key idea will be the same in most variations: looking at a right inverse of g to which the pigeonhole principle applies because it is an injection. The pigeonhole principle does not apply to g because initally we don't know that it is an injection.

7. The base case is $1/\sqrt{1} \le 2\sqrt{1}$ which is true because the left hand side equals 1 which is less than the right hand side which equals 2. For the induction step we have

$$\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \le 2\sqrt{k}$$

which implies

$$\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k+1}} \le 2\sqrt{k} + \frac{1}{\sqrt{k+1}}.$$

The induction step will be established if we can show that

$$2\sqrt{k} + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1}.$$

Multiplying both sides of the above equation by the positive number $\sqrt{k+1}$ we have

$$2\sqrt{k(k+1)}+1 \leq 2(k+1) \Leftrightarrow 2\sqrt{k(k+1)} \leq 2k+1 \Leftrightarrow 4k(k+1) \leq 4k^2+4k+1 \Leftrightarrow 4k^2+4k \leq 4k^2+4k+1$$

which is true. Reversing the above analysis, we prove the induction step.

8. In this case, it is much easier to find a closed-form (i.e., no ellipsis) expression for the sum on the left hand side and prove that the inequality holds for the closed-form expression. To find such an expression, you could try experimenting and coming up with your own induction hypothesis, but there is a more systematic way. Note that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Then

$$\frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \cdots + \frac{1}{n(n+1)} = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} \cdots - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}.$$

(You can prove that statement by induction if you like, but I am convinced by the above calculation.) Now it's easy to prove that $1 - 1/(n+1) < 1$, and the desired result follows.

9. If we apply the diagonal method to a list of rational numbers, there is no guarantee that the number constructed by modifying the diagonal entries will be a rational number. It will certainly be a real number, but it may be irrational, so the contradiction that makes the diagonal argument work cannot be achieved to prove that $\mathbb{Q}$ is uncountable.

10. (a) Suppose we have a complete list of the functions mapping $\mathbb{N} \to \mathbb{N}$: f_1, f_2, f_3, ... Consider the function $g : \mathbb{N} \to \mathbb{N}$ given by $g(n) = f_n(n) + 1$. (Make up a table of $f_m(n)$ to illustrate how g is formed from the diagonal.) Then g cannot be in the list of functions: $g \neq f_1$ because $g(1) \neq f_1(1)$; $g \neq f_2$ because $g(2) \neq f_2(2)$; and so on. Since g cannot be in the list, our assumption that the set of functions mapping $\mathbb{N} \to \mathbb{N}$ is countable must be wrong.

    (b) Each computable function can be written in terms of a computer program. Since the set of computer programs is countable, and the set of functions mapping $\mathbb{N} \to \mathbb{N}$ is uncountable by the above, there must be some function (in fact, many, many functions) which cannot be expressed in terms of a computer program. Most functions are not computable. Functions on the natural numbers are much more complicated than functions on finite sets.