MATH221-001 200530 Problem Set 7

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Due: Friday, November 25, 2005

1. Let P be the set of all people who have ever lived. Please fill in Table 1 with yes/no answers identifying the properties of various relations on $P \times P$. Here 'brother' means full brother, i.e., same mother and father; 'sibling' means full sibling, i.e., same mother and father.

xRy	Reflexive?	Symmetric?	Transitive?	Equivalence Relation?
x is y's brother				
x is y's sibling				
x is y's older sibling				
x and y have same last name				

Table 1: Properties of various human relations

2. Show that the following sets are countable:
   (a) The set of natural numbers which either one more or three more than a multiple of 7.
   (b) The set of all pairs of integers $(z_1, z_2) \in \mathbb{Z} \times \mathbb{Z}$.
3. (a) Show that $a = 1 + \sqrt{2}$ and $b = 1 - \sqrt{2}$ are irrational.
   (b) Show that $a + b$ and ab are rational.
   (c) Show that $a - b$ and a/b are irrational.
4. Show how the previous problem allows you to conclude that the sum and product of two irrational numbers are sometimes rational, sometimes irrational.
5. Show that the function $f : \mathbb{Q} \to \mathbb{Q}$ defined by $f(x) = 1 - \dfrac{1}{x}$ is an injection. Why is it not a surjection? Find a left inverse for f.
6. Show that if X is a finite set and $g : X \to X$ is a surjection then it must also be an injection.
7. Show that $\dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}} \le 2\sqrt{n}$ for all $n \in \mathbb{N}$.
8. Prove that $\dfrac{1}{1(2)} + \dfrac{1}{2(3)} + \cdots + \dfrac{1}{n(n+1)} < 1$ for all $n \in \mathbb{N}$. (Hint: it is sometimes easier to prove more than what is requested; this is the *inventor's paradox*.)
9. Why can the diagonal method not be used to show that $\mathbb{Q}$ is uncountable?
10. Let S be the set of functions which map $\mathbb{N}$ to $\mathbb{N}$.
    (a) Use the diagonal method to show that S is uncountable.
    (b) It seems reasonable that the set of computer programs is countable; after all, computer programs are stored in files, and the set of files is countable because each file can be associated with a (possibly very large) number encoding the file in binary. Show that this means that there are functions $f : \mathbb{N} \to \mathbb{N}$ which cannot be computed by any computer program.