MATH221-001 200530 Problem Set 8 Solutions DRAFT

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1. Since the question is slightly asymmetric the solution will be slightly asymmetric too; dividing into cases where the friends box contains 5 people or the strangers box contains 5 people doesn't seem to work. See Figures 1 and 2 for the worst case scenarios. In the figures, mutual friendship between two people is indicated by a blue edge and non-friendship by a red edge, and gray edges indicate a situation that reduces to a previous problem. (Many edges irrelevant to the argument are missing.) It turns out that in the worst case for Alice's friends, there are enough people (4) in the strangers box to draw the required conclusion, and in the worst case for Alices' strangers, there are enough people in the friends box to draw the required conclusion.

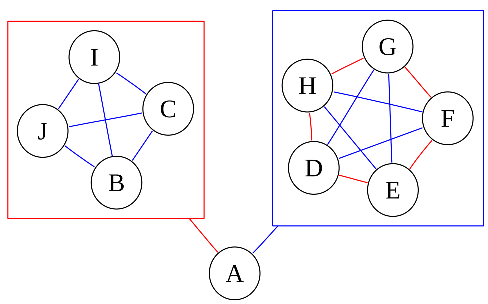

Figure 1: Worst case 1: In Alice's 5 friends, no 3 mutual friends, no 3 mutual strangers

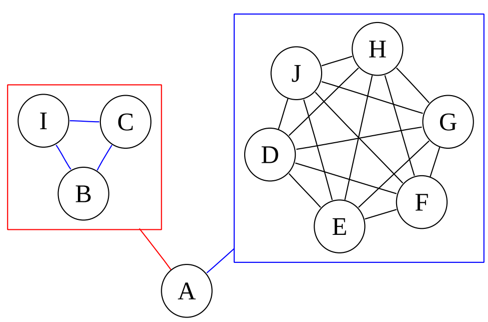

Figure 2: Worst case 2: In Alice's 3 strangers, no 4 mutual friends, no 2 mutual strangers

Pick any one of the ten people, Alice, say. Put all of the people who are strangers to Alice into a box, and all the people who are friends to Alice into another box. Either the strangers box contains 4 or more people or the friends box contains 6 or more people.

If the strangers box contains 4 or more people (e.g., Figure 1, then either some 2 of them (Bob and Carol, say) are strangers to each other in which case Alice, Bob, and Carol are 3 mutual strangers and we are done; or all individuals (at least 4) in the strangers box are mutual friends and we are again done.

On the other hand, if the friends box contains 6 or more people (e.g., Figure 2), then by the result in the textbook there must be either 3 mutual friends or 3 mutual strangers in the group of 6. If there is a set of 3 mutual strangers in the group of 6 we are done. If there is a set of 3 mutual friends in the group of 6 (Dawn, Edward, and Fred, say), then Alice, Dawn, Edward, and Fred is a set of 4 mutual friends and again we are done.

(This is an example of *Ramsey theory.* The minimum size of a group in which there must be either 3 mutual friends or 4 mutual strangers is actually 9, not 10, but the result is harder to prove.)

2. For this problem we use ordered selections with replacement. The number of different letter portions of the plate is equal to the number of functions mapping $\mathbb{N}_3$ to the alphabet, i.e., 26^3. The number of different number portions of the plate is equal to the number of functions mapping $\mathbb{N}_3$ to the numerical digits, i.e., 10^3. The total number of plates is therefore $26^3 \times 10^3 = 17,576,000$.

3. For this problem we use unordered selections with replacement. The number of choices for letters is $\binom{26+3-1}{3} = \frac{28 \times 27 \times 26}{3 \times 2 \times 1} = 3,276$. The number of choices for numbers is $\binom{10+3-1}{3} = \frac{12 \times 11 \times 10}{3 \times 2 \times 1} = 220$. Therefore the total number of choices in this problem is $3,276 \times 220 = 720,720$.

4. For this problem we use ordered selections without replacement. The number of selections of letters is $26 \times 25 \times 24 = 15,600$. The number of selections of numbers is $10 \times 9 \times 8 = 720$. The total number of selections is $11,232,000$.

5. For this problem we use unordered selections without replacement. The number of unordered slections without replacement for the letter portion of the plate is $\binom{26}{3} = \frac{26 \times 25 \times 24}{3 \times 2 \times 1} = 2,600$. The number of selections for the number portion is $\binom{10}{3} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120$. The total number of selections is $312,000$. There are $3 \times 2 \times 1 = 6$ ways of arranging selection of letters and 6 ways of arranging the selection of numbers; note that $312,000 \times 6 \times 6 = 11,232,000$, just as one would hope.

6. When n changes, the majority of the terms on the left hand side do not change; therefore induction on n seems like the easiest way to establish this identity. The identity can also be established by a counting argument, and I'm sure in many other ways.

   (a) **Induction argument.** For the base, let $n = 0$. Then $\binom{s-1}{0} = 1 = \binom{s+0}{0}$, establishing the base case. For the induction step, assume that for some k,

   $$\binom{s-1}{0} + \binom{s}{1} + \binom{s+1}{2} + \cdots + \binom{s+k-2}{k-1} + \binom{s+k-1}{k} = \binom{s+k}{k}.$$

   Then adding the next term in the series to both sides of the above equation,

   $$\binom{s-1}{0} + \binom{s}{1} + \binom{s+1}{2} + \cdots + \binom{s+k-2}{k-1} + \binom{s+k-1}{k} + \binom{s+k}{k+1} = \binom{s+k}{k} + \binom{s+k}{k+1}.$$

   The expression on the right hand side of the above is equal to $\binom{s+k+1}{k+1}$ by Theorem 11.1.1. That establishes the induction step, and the result follows for all n by induction.

   (b) **Counting argument.** The number of choices of n elements from $\mathbb{N}_{s+n}$ is $\binom{s+n}{n}$. Classify the choices as follows: Class 0 contains the n-subsets which include 1, 2, ..., n and do not include $n+1$. Class 1 contains the n-subsets which include the first 1, 2, ..., $n-1$ and do not include

n. Class 2 contains the n-subsets which include 1, 2, ..., $n-2$ and do not include $n-1$. Class $n-1$ contains the n-subsets which include 1 and do not include 2. Class n contains the n-subsets which do not include 1. A moment's thought should convince you that all the n-subsets fall into one and only one of the above classes.

For example, if $s = 3$ and $n = 2$ and our set is $\mathbb{N}_{s+n} = \mathbb{N}_5$, class 0 contains only the set $\{1, 2\}$, class 1 contains the sets $\{1, 3\}$, $\{1, 4\}$, and $\{1, 5\}$, and class 2 contains the rest of the 2-sets $\{2, 3\}$, $\{2, 4\}$, $\{2, 5\}$, $\{3, 4\}$, $\{3, 5\}$, and $\{4, 5\}$. We have classified the all 10 of the 2-subsets of $\mathbb{N}_5$ into three classes.

The number of subsets in class 0 is $1 = \binom{s-1}{0}$. The number of subsets in class 1 is $\binom{s}{1}$ because after selecting the first $n-1$ elements and discarding the element in position n we are free to select one more element from the remaining s possibilities. The number of subsets in class 2 is $\binom{s+1}{2}$ because after selecting the first $n-2$ elements and discarding the element in position $n-1$ we are free to choose two more elements from the remaining $s+1$ possibilities. Continuing in this way, we obtain each of the terms on the left hand side of the identity.

(c) **Binomial theorem.** We need some practice with this method. Unfortunately, the application of the binomial theorem to this problem is a little more difficult than average. See the textbook section on the binomial theorem for some easier questions.

Note that $\binom{s+n}{n}$ is the coefficient of x^n in the expansion of $(1+x)^{s+n}$, while the terms on the left hand side correspond to the binomial coefficients in Table 1. Since it is difficult to compare

Term	Coefficient of	In Expansion of
$\binom{s-1}{0}$	x^0	$(1+x)^{s-1}$
$\binom{s}{1}$	x^1	$(1+x)^{s}$
$\binom{s+1}{2}$	x^2	$(1+x)^{s+1}$
...	...	...
$\binom{s+n-2}{n-1}$	x^{n-1}	$(1+x)^{s+n-2}$
$\binom{s+n-1}{n}$	x^n	$(1+x)^{s+n-1}$

Table 1: Terms appearing in given series and corresponding coefficients

coefficients of different powers of x, we can alter the algebraic expressions in the table so that we are comparing like powers of x. We do so by multiplying each algebraic expression in a row of the table through by the appropriate power of x. See Table 2. It follows that the sum of the binomial coefficients in the first column of Table 2 is the same as the coefficient of x^n in the algebraic expression formed by the sum of the third column of Table 2.

In order to sum the third column of the table, recall that the sum of a geometric series with initial term a and common ratio r is given by the formula

$$a + ar + ar^2 + \cdots + ar^n = a\frac{r^{n+1} - 1}{r - 1}.$$

Term	Coefficient of	In Expansion of
$\binom{s-1}{0}$	x^n	$x^n(1+x)^{s-1}$
$\binom{s}{1}$	x^n	$x^{n-1}(1+x)^s$
$\binom{s+1}{2}$	x^n	$x^{n-2}(1+x)^{s+1}$
...	...	...
$\binom{s+n-2}{n-1}$	x^n	$x(1+x)^{s+n-2}$
$\binom{s+n-1}{n}$	x^n	$(1+x)^{s+n-1}$

Table 2: Terms appearing in given series and modified coefficients

In the case of the third column we have $a = x^n(1+x)^{s-1}$ and $r = (1+x)/x$, so the sum is

$$\begin{aligned}
S(x) &= x^n(1+x)^{s-1} + x^{n-1}(1+x)^s + x^{n-2}(1+x)^{s+1} \cdots + x^0(1+x)^{s+n-1} \\
&= x^n(1+x)^{s-1}\frac{((1+x)/x)^{n+1} - 1}{((1+x)/x) - 1} \\
&= (1+x)^{s-1}\left((1+x)^{n+1} - x^{n+1}\right) \\
&= (1+x)^{s+n} - x^{n+1}(1+x)^{s-1}.
\end{aligned}$$

There are no terms of the form Cx^n in the expression $x^{n+1}(1+x)^{s-1}$ because all terms have x to the power $n+1$ or greater, so the coefficient of x^n in $S(x)$ is the same as the coefficient of x^n in $(1+x)^{s+n}$, which is what we wanted to prove.

7. When n changes, the majority of the terms on the left hand side do not change, so this identity is also a good candidate for induction. However, we need more practice with the binomial theorem. We can also use a counting argument.

   (a) **Binomial theorem.** The right hand side of the identity is the coefficient of x^{m+1} in the expansion of $R(x) = (1+x)^{n+1}$. The left hand side of the identity is the coefficient of x^m in the expansion of $(1+x)^m + (1+x)^{m+1} + \cdots + (1+x)^n$, or alternatively the coefficient of x^{m+1} in $S(x) = x(1+x)^m + x(1+x)^{m+1} + \cdots + x(1+x)^n$. (Draw up tables similar to those in the previous problem to see why.) However, summing the geometric series $S(x)$,

$$\begin{aligned}
S(x) &= x(1+x)^m + x(1+x)^{m+1} + \cdots + x(1+x)^n \\
&= x(1+x)^m\left(1 + (1+x) + (1+x)^2 + \cdots + (1+x)^{n-m}\right) \\
&= x(1+x)^m\left(\frac{(1+x)^{n-m+1} - 1}{(1+x) - 1}\right) \\
&= x(1+x)^m\left(\frac{(1+x)^{n-m+1} - 1}{x}\right) \\
&= (1+x)^{n+1} - (1+x)^m.
\end{aligned}$$

   Since $R(x)$ and $S(x)$ differ only in terms of order m or less, the coefficient of x^{m+1} must be the same in both cases which establishes the identity.

   (b) **Counting argument.** The right hand side of the identity is the number of choices of $m+1$ numbers from an $\mathbb{N}_{n+1}$. Classify the choices according to the following scheme: Class m contains subsets which include $m+1$ and no number greater than $m+1$. Class $m+1$ contains subsets which include $m+2$ and no number greater than $m+2$. Class n contains subsets which include $n+1$. Every $m+1$-subset is in one of the given classes (why?), and no $m+1$-subset is in more than one of the given classes (why?). The size of each of the classes gives a term of the series on the left hand side of the given identity.

(c) **Induction.** It's best to perform induction on n. The induction step is established using the identity

$$\binom{k+1}{m} + \binom{k+1}{m+1} = \binom{k+2}{m+1},$$

i.e., Theorem 11.1.1.

8. Since $x^2+2xy+y^2$, it follows that $(x^2+2xy+y^2)^3 = ((x+y)^2)^3 = (x+y)^6$. By the binomial theorem, the coefficient of x^2y^4 is $\binom{6}{4} = 15$.

9. This is very hard. See me if you want more details.

10. See Figures 3 and 4. We need to form a large collection of subsets of $\mathbb{N}_n$ which does not contain any subsets which can be connected by a series of lines moving upward in the diagram. For example, $\{1\}$ and $\{12\}$ can be connected by a series of lines moving upward in Figure 3 so we cannot include both $\{1\}$ and $\{1,2\}$ in our collection of subsets of $\mathbb{N}_3$. Similarly $\{2\}$ and $\{2,3,4\}$ can be connected by lines moving upward in Figure 4 so they cannot both be included in our collection of subsets of $\mathbb{N}_4$. (A series of subsets that can be connected by lines moving upward is called a *chain*. What we want is called an *antichain*.) This gives us the idea that we can form a collection of subsets by using sets

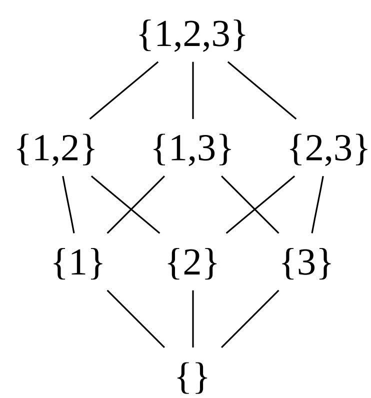

Figure 3: Subset lattice for $\mathbb{N}_3 = \{1,2,3\}$

only at the same level; clearly we can't connect two sets in such a collection by lines moving upward. Let's pick the widest level in the diagrams; for n even there is only one choice, the level containing the n^*-subsets of $\mathbb{N}_n$. For n odd there are two choices, the level containing the n^*-subsets of $\mathbb{N}_n$ or the level right above it; we'll take the former. In summary, we propose that a collection of subsets with the necessary properties is given by the n^*-subsets.

To prove the assertion, note that there are $\binom{n}{n^*}$ subsets in the collection we have formed. Furthermore, no two distinct subsets A and B in such a collection have the property that one is contained in the other. Assume for the sake of argument that $A \subset B$. Then $A \cup B = A$. But since A and B are distinct there is an element $b \in B$ such that $b \notin A$. Therefore $|A| < |B|$ which contradicts the fact that A and B are both n^*-sets. Our assumption that $A \subset B$ must be wrong; therefore no two subsets in our collection form a chain, and our collection is the required antichain.

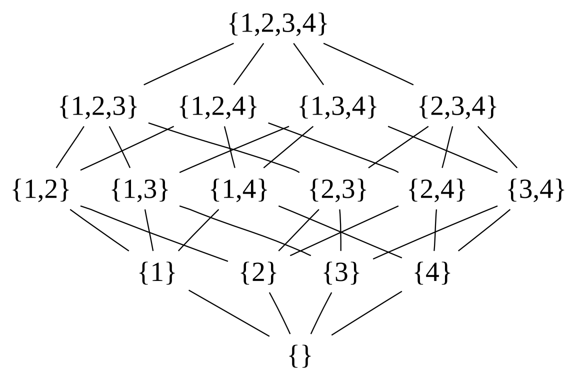

Figure 4: Subset lattice for $\mathbb{N}_4 = \{1, 2, 3, 4\}$

(Hard problem: show that the above is the best we can do. If you want a hint, look up *Sperner's Lemma*.)