MATH221-001 200630 Group Work Assignment 2: Infinite Sets

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Please try to solve as many of the following problems as you can with your group. At the end, hand in one set of answers for everyone in the group. Make sure that everyone in your group puts their name on the answer pages. The problems will be marked, and everyone in the group will get the same grade. You problably won't have time do do all the problems, so select the ones you can do best.

1. **Examples of infinite sets.** For each of the following sets S, show that S is infinite by constructing a bijection from S to $\mathbb{N}$.

   (a) $S = \{n \in \mathbb{N} : n \geq 23\}$

   (b) $S = \{n \in \mathbb{N} : n \text{ is a multiple of } 5\}$

   (c) $S = \{n \in \mathbb{N} : n \text{ ends in the digits } 00\}$

   (d) $S = \{n \in \mathbb{N} : n \text{ is a perfect square}\}$

2. **Countable sets.** A set S is called *countable* if there is a bijection $f : S \to \mathbb{N}$. Show that

   (a) $\mathbb{N}$ is countable.

   (b) Any subset of $\mathbb{N}$ is countable or finite.

   (c) The intersection of two countable sets is countable or finite.

   (d) The union of two countable sets is countable.

3. **Functions on finite and infinite sets.**

   (a) Suppose S is an infinite set. Show that there is an injection $f : S \to S$ which is not a surjection.

   (b) Suppose S is an infinite set. Show that there is a surjection $g : S \to S$ which is not an injection.

   (c) Suppose S is a finite set. Show that any injection $f : S \to S$ must be a bijection.

   (d) Suppose S is a finite set. Show that any surjection $g : S \to S$ must be a bijection.

4. **Primes.**

   (a) Explain, in your own words, Euclid's proof that the set of primes is infinite. (See p.52 of the textbook.)

   (b) An upper prime is a prime of the form $4n+3$ where n is a natural number. For example, 3, 7, and 11 are all upper primes, while 2, 5, and 13 are not upper primes. Adapt Euclid's proof to show that the set of upper primes is infinite.

   (c) Show that there are infinitely many primes of the form $6n+5$.

   (d) A lower prime is a prime of the form $4n+1$ where n is a natural number. For example, 5, 13, and 17 are lower primes. Explain why Euclid's proof cannot be adapted to show that the set of lower primes is infinite.

5. **The infinite hotel.**

(a) Suppose we want to accommodate one more guest in an already-full infinite hotel. Give an exact formula which describes to each guest to which room he or she should move.

(b) Suppose we want to accommodate one hundred more guests in an already-full infinite hotel. Give an exact formula which describes to each guest to which room he or she should move.

(c) Suppose we want to accommodate all the guests from one infinite hotel in another already-full infinite hotel. Give an exact formula which describes to each guest to which room he or she should move.

(d) Suppose we want to accommodate all the guests from an infinite chain of infinite hotels into another empty infinite hotel. Give an exact formula which describes to each guest to which room he or she should move. (You can base your formula on the system I described in the lecture, or you can devise a new system. It's OK if you don't use all the rooms.)

(e) Suppose we have an infinite number of cities, each with an infinite chain of infinite hotels. We want to accommodate all the guests from every hotel from every chain from every city in a single (empty) infinite hotel. Give an exact formula which tells every individual in any of those rooms to which room he or she should move in the empty infinite hotel so that no two individuals get the same room. (Hint: use the answer to the previous question.)