MATH221-001 200630 Group Work Assignment 3 Solutions DRAFT

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December 4, 2006

1. (a) R is not reflexive (no one is their own mother, e.g., you are not your own mother), not symmetric (if x is y's mother then x was born before y so y could not be x's mother; for a specific example pick any two people in the world), and not transitive: pick a grandmother x, mother y, and daughter z that you know. Then x is y's mother, y is z's mother, but x is not z's mother.

   (b) Here R is not reflexive (no one is their own ancestor), not symmetric (same reason as in previous question), but is transitive.

   (c) Reflexive, symmetric, and transitive, so it is an equivalence relation. The equivalence classes are sets of people all with the same last name; we could name those sets, e.g., Jones = { Al Jones, Fred Jones, ...}, Smith = { Jane Smith, Adam Smith, ...}.

   (d) If we assume monogamy marriage is symmetric but not reflexive nor transitive (take any married couple x and y; x is married to y and y is married to x but x is not married to x). If we assume polygamy we have to define better exactly what we mean by marriage.

   (e) Equivalence relation. The equivalence classes are: the set of all people born on January 1, the set of all people born on January 2, ..., the set of all people born on February 29, ..., the set of all people born on December 31.

2. (a) Reflexive because x is a multiple of x for any natural number x. Not symmetric because, e.g., 6 is a multiple of 3 but 3 is not a multiple of 6. Transitive by a simple argument: if x is a multiple of y and y is a multiple of z then $x = ky$, $y = nz$, $x = k(nz) = (kn)z$ and x is a multiple of z.

   (b) Not reflexive because, e.g., $2R2$ is false: $2 + 2$ is not a multiple of 3. Symmetric because xRy implies $x+y$ is a multiple of 3 implies $y+x$ is a multiple of 3 implies yRx. Not transitive because e.g. $2R4$ and $4R2$ but $2R2$ is false.

   (c) Equivalence relation. Reflexive because xRx is equivalent to $x + 2x = 3x$ is a multiple of 3 which is always true. Symmetric because xRy implies $x+2y$ is a multiple of 3 implies $3x+3y-(x+2y)$ is a multiple of 3 implies $2x + y$ is a multiple of 3 implies yRx. Transitive because xRy and yRz imply $x+2y+y+2z$ is a multiple of 3 implies $x+2z$ is a multiple of 3 implies xRz. The equivalence classes are all numbers such that $xR1$, i.e., all numbers which are 1 more than a multiple of 3; all numbers such that $xR2$, i.e., all numbers which are 2 more than a multiple of 3; and all numbers such that $xR3$, i.e., all numbers which are multiples of 3. (Can you prove that those are good descriptions of the equivalence classes, and are all the equivalence classes?)

   (d) The relation is reflexive because xRx because $x = 1x$ for all x. The relation is not symmetric because $130R13$ but $13R130$ is false. The relation is transitive because xRy and yRz implies $x = n_1y$ and $y = n_2z$ where n_1, n_2 are powers of 10 implies $x = n_1n_2z$, and n_1n_2 is a power of 10.

   (e) Equivalence relation. The relation is reflexive for the same reason as above. The relation is symmetric because if xRy then $x = ny$ or $y = nx$ which implies $y = nx$ or $x = ny$ which implies yRx. The relation is transitive as we can see by writing xRy if and only if $n_1x = n_2y$ for n_1, n_2 some multiples of 10. Then xRy and yRz implies $n_1x = n_2y$ and $n_3y = n_4z$ implies $n_1n_3x = n_2n_3y = n_2n_4z$ so $N_1x = N_2z$ where $N_1 = n_1n_3$ and $N_2 = n_2n_4$ are multiples of 10. The equivalence classes consist of all numbers which are powers of 10 times a number which is not divisible by 10, i.e., $\{1, 10, 100, 1000, \ldots\}$, $\{2, 20, 200, 2000, \ldots\}$, $\ldots$, $\{13, 130, 1300, 13000, \ldots\}$, $\ldots$. There is an infinite number of equivalence classes.

(f) Equivalence relation. The equivalence classes are numbers ending in 00, numbers ending in 01, $\ldots$, numbers ending in 99.

3. 1(a): reflexive, not symmetric, not transitive, etc., etc.

4. Reflexive, not symmetric, and weirdly, not transitive. You will need to draw up tables of each of the 36 possible rolls for each of the 3 pairs of dice to see why. If you don't believe me, let's play dice.

5. No, consider for example the relation R on $\mathbb{N}$ given by xRy if and only if x and y are both not 1. The relation is symmetric (why?) and transitive (why?) but not reflexive because it is not the case that $1R1$. Can you construct other examples? Given an equivalence relation can you construct a new relation which is symmetric and transitive but not reflexive?

6. (a) The relation $\leq$ on $\mathbb{N}$ is total because $x \leq x + 1$ for all x.
   (b) The relation $>$ on $\mathbb{N}$ is not total because it is never true that $1 > x$.
   (c) The relation $\leq$ on $\mathbb{N}$ is both transitive and total.
   (d) The relation $<$ on $\mathbb{N}$ is transitive, total, and irreflexive. (Why?)
   (e) Let S be the set on which we have a transitive, total, and irreflexive relation. Pick an element $s_1 \in S$. Then by the total property there must be an element s_2 such that $s_1 R s_2$. Again by the total property there must be an element s_3 such that $s_2 R s_3$, and we can continue by induction finding elements s_n such that $s_k R s_{k+1}$ for all $k \in \mathbb{N}$. Note that if $m < n$ then $s_m R s_n$ by transitivity (give a detailed argument). It follows that $s_m \neq s_n$ for any $m, n \in \mathbb{N}$, $m \neq n$, for otherwise (assuming $m < n$; the other case is similar) we would have $s_m R s_n$ and $s_m = s_n$ which contradicts irreflexivity. It follows that the set $\{s_n : n \in \mathbb{N}\}$ is infinite, so S must be infinite.