MATH221-001 200630 Midterm Test 1 Solutions DRAFT

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December 6, 2006

1. (a) See Table 1 for the truth table for the given expressions. (Ignore the last column for now.) Since the columns under $\neg(\neg p \vee q)$ and $\neg q \wedge p$ are the same, the expressions are logically equivalent.

p	q	$\neg p$	$\neg p \vee q$	$\neg(\neg p \vee q)$	$\neg q$	p	$\neg q \wedge p$	$\neg q \wedge p \Leftrightarrow \neg(\neg p \vee q)$
F	F	T	T	F	T	F	F	T
F	T	T	T	F	F	F	F	T
T	F	F	F	T	T	T	T	T
T	T	F	T	F	F	T	F	T

Table 1: Truth table for $\neg q \wedge p \Leftrightarrow \neg(\neg p \vee q)$

   (b) Now refer to the last column in Table 1 Since the last column is always true, the expression is a tautology.

   (c) Let $p(x)$ be the statement $x \in A$ and $q(x)$ be the statement $x \in B$. Then the given set relationship can be written

$$\forall x \in U : x \in (B^c \cap A) \Leftrightarrow x \in (A^c \cup B)^c.$$

   Expanding using the definitions of set union, complement, and intersection, we have $x \in (B^c \cap A)$ is equivalent to $x \in B^c \wedge x \in A$ which is equivalent to $\neg(x \in B) \wedge (x \in A)$, i.e., $\neg q(x) \wedge p(x)$. Therefore the statement above is equivalent to

$$\forall x \in U : \neg q(x) \wedge p(x) \Leftrightarrow \neg(\neg p(x) \vee q(x));$$

   however, the statement $\neg q(x) \wedge p(x) \Leftrightarrow \neg(\neg p(x) \vee q(x))$ is always true by the logical analysis in part (b), so the given set relationship is true.

2. Let r be the statement "it is raining", s the statement "it is snowing", l the statement "it is cloudy", and c the statement "it is cold". The given statement can be written

$$(r \vee s) \Rightarrow (l \wedge c).$$

   The negation is

$$\neg((r \vee s) \Rightarrow (l \wedge c)).$$

   To negate an implication, it is best to use the identity $P \Rightarrow Q \equiv \neg P \vee Q$ which means that

$$\neg((r \vee s) \Rightarrow (l \wedge c)) \equiv \neg(\neg(r \vee s) \vee (l \wedge c)).$$

   Now we can use "DeMorgan's Laws" and the double negative identity to obtain

$$\neg((r \vee s) \Rightarrow (l \wedge c)) \equiv \neg(\neg(r \vee s)) \wedge \neg(l \wedge c) \equiv (r \vee s) \wedge \neg(l \wedge c)$$

   which translates into words as "It is raining or snowing, but it is not the case that it is cloudy and cold." Going to a further level of detail, if you wish, we have

$$\neg((r \vee s) \Rightarrow (l \wedge c)) \equiv \neg(\neg(r \vee s)) \wedge \neg(l \wedge c) \equiv (r \vee s) \wedge \neg(l \wedge c) \equiv (r \vee s) \wedge (\neg l \vee \neg c),$$

   i.e., "It is raining or snowing, and it is not cloudy or not cold."

3. **Long, straightforward solution.** First we determine whether f is an injection. Suppose $f(n_1) = f(n_2)$. There are three possibilities: n_1, n_2 are both even, n_1, n_2 are both odd, or one of n_1, n_2 is even and the other odd. In the first case we have $f(n_1) = n_1 - 1$, $f(n_2) = n_2 - 1$, and since the values are equal, $n_1 - 1 = n_2 - 1$ which implies $n_1 = n_2$. In the second case we have $f(n_1) = f(n_2)$ implies $n_1 + 1 = n_2 + 1$ which implies $n_1 = n_2$. In the third case we have (say) n_1 even, n_2 odd so $f(n_1)$ is odd, $f(n_2)$ is even, and $f(n_1) = f(n_2)$ means an odd is equal to an even, impossible. In all possible cases, we have $f(n_1) = f(n_2)$ implies $n_1 = n_2$ so the function is an injection.

   Next we determine whether f is a surjection, i.e., given $n \in \mathbb{N}$ we want to find $m \in \mathbb{N}$ such that $f(m) = n$. If n is even, let $m = n - 1$. (The number m is always a natural number because if n is even and natural it must be at least 2.) Then m is odd so therefore $f(m) = m + 1 = (n - 1) + 1 = n$. On the other hand, if n is odd, let $m = n + 1$. Then m is even, so $f(m) = m - 1 = (n + 1) - 1 = n$. In either case, given n we have found m such that $f(m) = n$, so the function is a surjection.

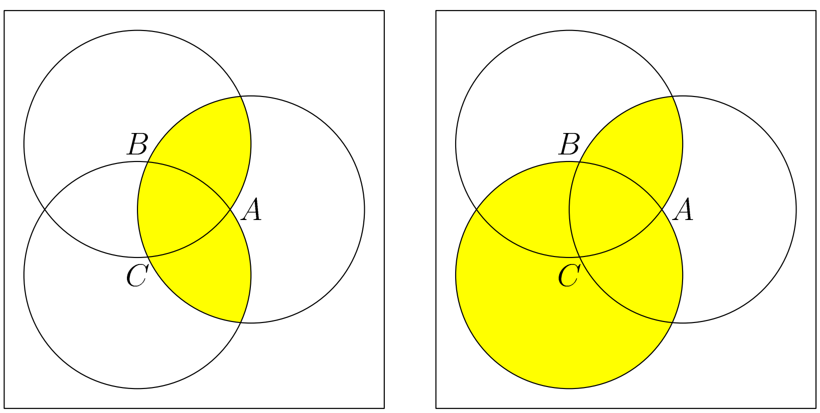
   Since it is both an injection and a surjection, it is a bijection.

   Short, elegant solution. Note that if n is even, then $n-1$ is odd so $f(f(n)) = f(n-1) = n-1+1 = n$. On the other hand, if n is odd, then $n+1$ is even so $f(f(n)) = f(n+1) = n+1-1 = n$. In any case, $f(f(n)) = n$, so $f \circ f = id$, the identity function. It follows that f has an inverse (namely f), so f must be a bijection.

4. (a) We argue by contradiction. Suppose $g \circ f$ is a surjection, but g is not a surjection. Since g is not a surjection, there is an element $s \in S$ such that $g(t) \neq s$ for all $t \in S$. Since $g \circ f$ is a surjection there is $u \in S$ such that $g \circ f(u) = s$. But then $g(f(u)) = s$, i.e., $g(t) = s$ where $t = f(u)$, contradicting the fact that $g(t) \neq s$ for all $t \in S$. Our assumptions lead to a contradiction so they must be false. Therefore $g \circ f$ is not a surjection.

   (b) Let $S = \mathbb{N}$, $g(n) = n + 1$, and $f(n) = n - 1$. Actually, that f doesn't work because $f(1)$ isn't in $\mathbb{N}$, so we redefine $f(1) = 1$ (or you could define $f(1) = 2$ or anything else; let's just stick with $f(1) = 1$, the simplest choice). Then g is not a surjection because $g(n) = 1$ is never true. However, f is a surjection (why?). Also, $f \circ g = f(n + 1) = (n + 1) - 1 = n$ for all $n \in \mathbb{N}$ so $f \circ g = id$ is a surjection.

5. Drawing a Venn diagram will help you understand the situation better. See Figure 1. You will see



(a) $A \cap (B \cup C)$ (b) $(A \cap B) \cup C)$

Figure 1: Venn diagrams for two sets

that the two regions are not the same, and in fact will give you an idea of how to construct a counter-example. In order to construct a counter-example, all we need are sets A, B, and C with some element in a yellow region in Figure 1(b) that is not in a yellow region in Figure 1(a). The easiest example is to find sets with an element e in C but not in A or B; we can do so, for example, by letting $A = \varnothing$, $B = \varnothing$, $C = \{e\}$. Then verifying that our example works, $B \cup C = \{e\}$, $A \cap (B \cup C) = \varnothing$, $A \cap B = \varnothing$, and $(A \cap B) \cup C = \{e\} \neq A \cap (B \cup C)$.

6. This is a hard problem. Now that we know modular arithmetic, though, we can dispense with it right away. The restriction of g to odd numbers is $n^2 + 3$, which is increasing, so it's impossible to have $g(n_1) = g(n_2)$ for two distinct odd numbers. Similarly, the restriction of g to even numbers is $4n + 8$, which is increasing, so it's impossible to have $g(n_1) = g(n_2)$ for two distinct even numbers. The only remaining case is if n_1 is odd and n_2 is even (or vice versa, in which case you can switch the indices). This is difficult because g is not increasing (graph it if you don't believe me) so we can't use information about the relative sizes of $g(n_1)$ and $g(n_2)$. We do, however, know that n_1 odd implies $n_1^2 \equiv 1 \mod 8$ implies $n^2 + 3 \equiv 4 \mod 8$; furthermore, n_2 even implies $4n_2 \equiv 0 \mod 8$ (why) implies $4n_2 + 8 \equiv 0 \mod 8$. So $g(n_1) \not\equiv g(n_2) \mod 8$, so $g(n_1)$ and $g(n_2)$ cannot possibly be equal. In all cases, if $n_1 \neq n_2$, we have $g(n_1) \neq g(n_2)$, so g is an injection.

7. This is another hard problem, similar to the final problem on the sample midterm.

   Since S is finite, it follows that given any $s \in S$, the sequence $s, f(s), f^2(s), f^3(s), \ldots$ must have a repeated element; say $f^m(s) = f^n(s)$ where $m < n$. Since f is a bijection, it is invertible; multiplying the last equation by $(f^{-1})^m$ we have $(f^{-1})^m(f^m(s)) = (f^{-1})^m(f^n(s))$ which implies $s = f^{n-m}(s)$, i.e., for each element $s \in S$ there is a number k_s such that $f^{k_s}(s) = s$.

   Now, multiply all those k_s all together to get a number

   $$k = k_1 \times k_2 \times \cdots \times k_s.$$

   Then $f^k(1) = f^{k_1}(f^{k_1}(\cdots f^{k_1}(1)\cdots)) = 1$, where the number of f^{k_1} in the above is k/k_1. Similarly $f^k(2) = 2$, etc., so f^k is the identity function, as required.

   This can fail on an infinite set; for example, let $S = \mathbb{Z}$ and $f(s) = s + 1$. Then $f^k(s) = s + k$ is never the identity function.