MATH221-001 200630 Midterm Test 1 Solutions DRAFT

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1. (a) Arrange the numbers as in Table 1. There are 5 rows of 11 pigeonholes in the table, so after

1 12	2 13	3 14	4 15	5 16	6 17	7 18	8 19	9 20	10 21	11 22
23 34	24 35	25 36	26 37	27 38	28 39	29 40	30 41	31 42	32 43	33 44
45 56	46 57	47 58	48 59	49 60	50 61	51 62	52 63	53 64	54 65	55 66
67 78	68 79	69 80	70 81	71 82	72 83	73 84	74 85	75 86	76 87	77 88
89 100	90	91	92	83	94	95	96	97	98	99

Table 1: Pigeonholes for numbers which differ by 11

choosing 56 numbers I must have two numbers from the same pigeonhole, so two of my numbers must differ by 11.

(b) If we are going to pick 55 numbers, none of which differ by 11, at least we must pick one number from each pigeonhole. But that condition alone is not enough! If we picked 12 and 23, for example, we would still have two numbers which differed by 11. So I will have to be more careful and spell out exactly what I'm doing. I will pick all the numbers in the first row, all the numbers in the third row, . . . , all the numbers in the ninth row. Then I'm sure that no two differ by 11 because of the "no man's land" strips around my selections. Note that I made an explicit selection of numbers, I didn't just say what I could do.

2. This question is actually absolutely straightforward. We have

$$\sum_{r=1}^{n}(3r^2+7r) = 3\sum_{r=1}^{n} r^2+7\sum_{r=1}^{n} r = \frac{3}{6}n(n+1)(2n+1)+\frac{7}{2}n(n+1) = \frac{1}{2}n(n+1)(2n+1+7) = n(n+1)(n+4).$$

There's no need to prove the formula by induction if you believe the two given formulas.

3. You could do this problem like the previous, but I asked specifically for a solution by induction. The base is $n = 1$, in which case the left hand side is

$$\sum_{r=1}^{1}(3r^2 + 5r) = 3(1)^2 + 5(1) = 8,$$

and the right hand side is

$$1(1 + 1)(1 + 3) = 1(2)(4) = 8;$$

the two sides agree so the formula is true when $n = 1$. Now suppose that the formula is true for some $n = k$. Then

$$\sum_{r=1}^{k}(3r^2 + 5r) = k(k + 1)(k + 3). \tag{1}$$

That is our *induction hypothesis*. What we want to do is to execute the induction step to prove that

$$\sum_{r=1}^{k+1}(3r^2+5r)=(k+1)((k+1)+1)((k+1)+3). \tag{2}$$

Note that at this point, equation (1) is assumed to be true but equation (2) is unproven, even though there is an equal sign in each. (I would prefer it if there were a different sign I could use in (2); sometimes I put a question mark over the equals sign to emphasize that it isn't yet proven, but that's hard for me to do with my computer typesetting system. Just keep in mind that it's our goal.)

One way to prove (2) would be to start with the left hand side and do algebra, etc., until we reach the right hand side, which is what we do. At each step, the next move is pretty clear:

$$\sum_{r=1}^{k+1}(3r^2+5r)=\sum_{r=1}^{k}(3r^2+5r)+3(k+1)^2+5(k+1)$$

by the definition of the summation symbol. But by the induction hypothesis we have

$$\sum_{r=1}^{k}(3r^2+5r)+3(k+1)^2+5(k+1)=k(k+1)(k+3)+3(k+1)^2+5(k+1).$$

Factoring the above expression we have

$$\begin{aligned}k(k+1)(k+3)+3(k+1)^2+5(k+1)&=(k+1)\left(k(k+3)+3(k+1)+5\right)=(k+1)(k^2+6k+8)\\&=(k+1)(k+2)(k+4)=(k+1)((k+1)+1)((k+1)+3).\end{aligned}$$

Stringing together all of the above equalities, we see we have established (2) under the assumption that (1) holds. In other words, we have established the *induction step*

$$\sum_{r=1}^{k}(3r^2+5r)=k(k+1)(k+3)\implies\sum_{r=1}^{k+1}(3r^2+5r)=(k+1)((k+1)+1)((k+1)+3).$$

By the Principle of Mathematical Induction, it follows that

$$\sum_{r=1}^{n}(3r^2+5r)=n(n+1)(n+3)$$

is true for all $n\in\mathbb{N}$, and we are done.

4. Again, this is quite straightforward. The base is the case when $n=1$, in which case the left hand side $f_1^2=1^2=1$ and the right hand side f_1f_2 are equal, so the result is true for $n=1$. Suppose the induction hypothesis that the result is true for $n=$ some k:

$$f_1^2+f_2^2+\cdots+f_k^2=f_kf_{k+1}.$$

The induction step is to prove the next result

$$f_1^2+f_2^2+\cdots+f_k^2+f_{k+1}^2=f_{k+1}f_{k+2}$$

under the assumption that the induction hypothesis is true. We do so like this:

$$f_1^2+f_2^2+\cdots+f_k^2+f_{k+1}^2=(f_1^2+f_2^2+\cdots+f_k^2)+f_{k+1}^2=f_kf_{k+1}+f_{k+1}^2$$

by the induction hypothesis. Factoring the above expression we have

$$f_1^2+f_2^2+\cdots+f_k^2+f_{k+1}^2=f_{k+1}(f_k+f_{k+1}).$$

By the definition of the Fibonacci numbers,

$$f_1^2+f_2^2+\cdots+f_k^2+f_{k+1}^2=f_{k+1}f_{k+2},$$

which establishes the induction step. By the Principle of Induction we are done.

5. After checking a few cases, you will see that the base case $2^3 > 2(3) + 1$ is true, and should develop some sense that each result can be obtained from the previous, so we should use ordinary mathematical induction, not strong induction for this problem. For the induction hypothesis, we assume that

$$2^k > 2k + 1 \tag{3}$$

for some k in the set $\{3, 4, 5, \ldots\}$. Under that assumption we need to perform the induction step and show that the result

$$2^{k+1} > 2(k + 1) + 1 \tag{4}$$

is true. Let me once again emphasize that (3) is assumed to be true in the context of the induction step, so we can use it, but (4) is not yet established; it is our goal. Put a question mark over the inequality sign in (4) if you have trouble keeping that straight.

We can make the left hand side of (3) look more like the left hand side of the goal (4) by multiplying both sides by 2:

$$2^{k+1} > 4k + 2. \tag{5}$$

Now if we can show that

$$4k + 2 > 2(k + 1) + 1, \tag{6}$$

we can chain together inequalities (5) and (6) and finish the induction step.

Now we've got everything except (6). But that inequality is equivalent to

$$2k > 1$$

which is certainly true for every $k = 3, 4, 5, \ldots$. That completes the last link in the chain of reasoning which establishes the induction step. (You should probably re-write the above analysis in reverse order to make a seamless proof. Remember, analysis like the above is the way to solve problems and understand the situation, but a proof often goes in reverse order from the analysis.)

Since we have the base and the induction step, the result follows from the Principle of Induction.

6. To show a set S is countable, by definition (see the textbook) we have to find a bijection $f : S \to \mathbb{N}$. Let us first find a bijection $g : \mathbb{N} \to S$ which is often easier. In this case the obvious candidate is

$$g(n) = n(n + 1)(n + 3).$$

The function g is a surjection because S is defined simply as the image of $\mathbb{N}$ under g. To show that g is an injection is harder. Recall that if g is increasing, it must be an injection (the same principle is used to show that a function is one-to-one, i.e., and injection, in calculus, in case you are familiar with that). It certainly seems that g is increasing, but to prove it it is best to invoke question 3. We see that

$$g(n + 1) - g(n) = 3^(n + 1)^2 + 5(n + 1) > 0$$

so $g(n + 1) > g(n)$ for all n, so (technically by induction) g is increasing, so g is an injection.

It follows that g is a bijection, so it has an inverse function $g^{-1} : S \to \mathbb{N}$. Then the function $f = g^{-1}$ has all the required properties to show that S is countable.