MATH221-001 200630 Problem Set 1 Solutions DRAFT

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1. See Table 1 for the truth table for the given expressions. (You may find it helpful to add extra columns for q and p between the third and fourth columns and between the second last and last columns.) Since the last two columns are identical, the expressions are logically equivalent.

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$	$p \Leftrightarrow q$
F	F	T	T	T	T
F	T	T	F	F	F
T	F	F	T	F	F
T	T	T	T	T	T

Table 1: Truth table for $(p \Rightarrow q) \wedge (q \Rightarrow p)$ and $p \Leftrightarrow q$

2. See Table 2 for the truth table for the given expressions. Since the columns under $\neg(p \vee q) \wedge r$ and

p	q	r	$p \vee q$	$\neg(p \vee q)$	$\neg(p \vee q) \wedge r$	$\neg r$	$(p \vee q) \wedge \neg r$	$\neg((p \vee q) \wedge \neg r)$
F	F	F	F	T	F	T	F	T
F	F	T	F	T	T	F	F	T
F	T	F	T	F	F	T	T	F
F	T	T	T	F	F	F	F	T
T	F	F	T	F	F	T	T	F
T	F	T	T	F	F	F	F	T
T	T	F	T	F	F	T	T	F
T	T	T	T	F	F	F	F	T

Table 2: Truth table for $\neg(p \vee q) \wedge r$ and $\neg((p \vee q) \wedge \neg r)$

under $\neg((p \vee q) \wedge \neg r)$ differ, the two expressions are not logically equivalent. Specifically, they differ when all three of p, q, and r are false (and in many other combinations such as all true, etc.; but we only need one difference to show that the expressions are not logically equivalent).

To construct English sentences, let p, q, and r be any propositions which are false, for example $p =$ "Curious George is a hummingbird", $q =$ "2 is composite", and $r =$ "2 is a monkey". Then the first expression becomes, "It is not the case that either Curious George is a hummingbird or 2 is composite, and 2 is a monkey" (which is false). Similarly, the second sentence becomes, "It is not the case that: Curious George is a hummingbird or 2 is composite, and 2 is not a monkey" (which is true).

3. Draw up a truth table for the given statement as in Table 3. Note that a number of columns are repeated; that is just to facilitate checking. Note that every entry in the final column is True, so the statement is always true, i.e., it is a tautology.

4. To answer this question, we write the statement as an implication $p \Rightarrow q$ where $p =$ "Curious George is a monkey and 2 is a prime" and $q =$ "2 is a monkey". Then the converse is $q \Rightarrow p =$ "If 2 is a monkey

p	q	$\neg p$	q	$\neg p \Rightarrow q$	$\neg p$	$\neg q$	$\neg p \Rightarrow \neg q$	$(\neg p \Rightarrow q) \wedge (\neg \Rightarrow \neg q)$	p	$((\neg p \Rightarrow q) \wedge (\neg \Rightarrow \neg q)) \Rightarrow p$
F	F	T	F	F	T	T	F	F	T	T
F	T	T	T	T	T	F	F	F	F	T
T	F	F	F	T	F	T	T	T	T	T
T	T	F	T	T	F	F	T	T	T	T

Table 3: Truth table for $((\neg p \Rightarrow q) \wedge (\neg \Rightarrow \neg q)) \Rightarrow p$

then Curious George is a monkey and 2 is prime", the inverse is $\neg p \Rightarrow \neg q =$ "If Curious George is not a monkey or 2 is not prime, then 2 is not a monkey", and the contrapositive is $\neg q \Rightarrow \neg p =$ "If 2 is not a monkey, then Curious George is not a monkey or 2 is not prime". There is an infinite number of correct variations on each sentence. Also, note the lack of precision in English due to the lack of parentheses in spoken English. Logic was created partly to resolve the lack of precision in natural language.

5. Let $P =$ "Private schools implementing a voucher system will fail to provide equal educational opportunities across a community", $S =$ "They skim off the best students", $B =$ "They leave the poorer ones behind", $C =$ "They charge parents fees beyond what is paid for in private funds and so exclude children of poorer families". The latter "and" doesn't seem to be a logical 'and' to me, although one may be able to argue that it is; it seems more like "and so" could be modelled by an implication, but let's just avoid that debate entirely by leaving those phrases together as a group in our analysis. A more detailed analysis would derive a number of logical statements from a single English statement, including the implicit assumptions in the English statement.

   Anyway, now we can write the given statement in logical notation: $((S \wedge B) \vee C) \Rightarrow P$. To negate an implication we first rewrite the implication in terms of 'and', 'or', and 'not': $\neg((S \wedge B) \vee C) \vee P$. Now we negate the whole thing: $\neg(\neg((S \wedge B) \vee C) \vee P)$. It's usually better when converting back into English to remove as many brackets as possible. In this case we get $\neg(\neg((S \wedge B) \vee C) \wedge \neg P$ or $((S \wedge B) \wedge) \neg C \wedge P$. Converting back into English, we have something like "Private schools implementing a voucher system skim off the best students, leave the poorer ones behind, charge parents fees beyond what is paid for in private funds and so exclude the children of poorer families, and (yet) succeed in providing equal educational opportunities across a community." No agreeing with that, is there?

6. Suppose the man is a knave. Then his statement must be false. The only way for his statement (an implication) to be false is for the antecedent, "I am a knight", to be true and the consequent, "My wife is a knight", to be false. That would mean that the man must be a knight, which contradicts the assumption that he is a knave. Therefore the man must be a knight, so his statement is true. Since the antecedent of his statement is true, so must be the consequent, so his wife is a knight also. In summary, the man and his wife must both be knights.

7. These examples are based on "Symbolic Notation, Haddocks' Eyes and the Dog-Walking Ordinance" by Ernest Nagel, from *The World of Mathematics* edited by James R. Newman. Since the dog will be on a leash sometimes and not on a leash other times, and since the dog may sometimes be in the park and sometimes not, time is an important factor in most if not all of these questions. For example, in the first, we want to capture the idea that *when* the dog is brought in to the park, it must be on a leash. The following open statements will be used more than once: $D(x) =$ "x is a dog", $L(x,t) =$ "x is required to be on a leash at time t", $O(y,x) =$ "y is the owner of x", $P(x,t) =$ "x is in this park at time t", $R(y,x,t) =$ "y is required to keep x on a leash at time t.

   (a) $\forall x : \forall t : (D(x) \wedge$ x is brought into this park at time t $\Rightarrow L(x,t)$. (There's nothing to prevent an owner from removing the leash once the dog is in the park, though.)

   (b) This is similar to the first, with perhaps the added stylistic issue that the sign is not addressed to the owners.

(c) $\forall x : \forall y : \forall t : (D(x) \wedge O(y, x) \wedge P(y, t)) \Rightarrow L(y, x, t)$. This rule is consistent with the owner of a dog being in the park without the dog, the owner being nevertheless required to keep his absent dog on a leash.

(d) $\forall y : \forall t : \exists x : P(y, t) \Rightarrow (D(x) \wedge O(y, x) \wedge R(y, x, t))$. A necessary condition for being in the park at any time would be the possession of a dog which is on a leash at that time.

(e) $\forall x : \exists t : D(x) \Rightarrow (P(x, t) \wedge L(x, t))$. All dogs in the universe must at some point be walked in this park on a leash. Another interpretation may be $\forall x : \forall t : (D(x) \wedge$ x is on a leash at time t $\Rightarrow$ x is required to be on a leash at time t . A necessary condition for a dog being on a leash is that the animal be on a leash in the park.

(f) $\forall x : \forall t : D(x) \Rightarrow (P(x, t) \wedge L(x, t))$. This is even worse than the previous, since it requires that all dogs in the universe always be kept on a leash in the park.

A suggested alternative: "All dogs in this park must be kept on a leash". Can you find anything wrong with it? (I can't.)

8. (I will write a more detailed solution to this hard problem later.)

(a) Yes; for example, Brown and Carr could go out and Allen could stay in. Or Carr could go out by himself and Allen and Brown could stay in.

(b) Yes; you could demonstrate equivalence using truth tables, for example.

(c) Again, truth tables.

(d) The contrapositive of the statement is, "If it is not the case that if Allen goes out then Brown stays in, then it is not the case that Carr goes out". Since "if Allen goes out then Brown stays in" is false, "it is not the case that if Allen goes out then Brown stays in" is true; since the contrapositive statement is true, and the antecedent is true, it follows that the consequent is true, i.e., "it is not the case that Carr goes out" is true.

The paradox can be resolved by using the equivalence $p \Rightarrow q \equiv \neg p \vee q$. This paradox is really just a complicated way of showing that we must regard $p \Rightarrow q$ as meaning the same thing as $\neg p \vee q$.