MATH221-001 200630 Problem Set 2 Solutions DRAFT

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1. (a) See Table 1 for the truth table for the given expressions. Since the last column is always True, the expression is a tautology.

p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$	$\neg p$	q	$(\neg p \vee q)$	$\neg(p \wedge \neg q) \Leftrightarrow (\neg p \vee q)$
F	F	T	F	T	T	F	T	T
F	T	F	F	T	T	T	T	T
T	F	T	T	F	F	F	F	T
T	T	F	F	T	F	T	T	T

Table 1: Truth table for $\neg(p \wedge \neg q) \Leftrightarrow (\neg p \vee q)$

(b) See Table 2 for the truth table for the given expressions. Since the last column is always True, the expression is a tautology.

p	q	r	$\neg r$	$q \vee \neg r$	$\neg(q \vee \neg r)$	$p \wedge \neg(q \vee \vee \neg r)$	$\neg q$	$p \wedge \neg q$	$p \wedge \neg(q \vee \vee \neg r) \Rightarrow p \wedge \neg q$
F	F	F	T	T	F	F	T	F	T
F	F	T	F	F	T	F	T	F	T
F	T	F	T	T	F	F	F	F	T
F	T	T	F	T	F	F	F	F	T
T	F	F	T	T	F	F	T	T	T
T	F	T	F	F	T	T	T	T	T
T	T	F	T	T	F	F	F	F	T
T	T	T	F	T	F	F	F	F	T

Table 2: Truth table for $p \wedge \neg(q \vee \vee \neg r) \Rightarrow p \wedge \neg q$

2. Let $p(x)$ be the statement $x \in A$, $q(x)$ be the statement $x \in B$, and $r(x)$ be the statement $x \in C$. In the following, U is a universal set, without which the set complement operator X^c is not meaningful. (It's OK if you didn't think about the universal set this time around, but try to think of it from now on.)

(a) In this question we have to prove that

$$\forall x \in U : x \in (A \cap B^c)^c \Leftrightarrow x \in A^c \cup B.$$

Expanding the meaning of those set operations, we have to prove

$$\forall x \in U : \neg(x \in (A \cap B^c)) \Leftrightarrow x \in A^c \vee x \in B,$$

i.e.,

$$\forall x \in U : \neg((x \in A) \wedge (x \in B^c) \Leftrightarrow \neg(x \in A) \vee (x \in B),$$

i.e.,

$$\forall x \in U : \neg((x \in A) \wedge \neg(x \in B)) \Leftrightarrow \neg(x \in A) \vee (x \in B),$$

i.e.,

$$\forall x \in U : \neg(p(x) \wedge \neg q(x)) \Leftrightarrow \neg p(x) \vee q(x).$$

However, the latter statement is true (independent of x) by 1(a); it follows that the original statement, i.e., the set identity, is true.

(b) This is similar to the previous, except that we have to prove the statement

$$\forall x \in U : x \in (A \cap (B \cup C^c)^c) \Rightarrow x \in (A \cap B^c)$$

which, by an analysis like that of the previous question, is equivalent to the statement

$$\forall x \in U : p(x) \wedge \neg(q(x) \vee \neg r(x)) \Rightarrow p(x) \wedge \neg q(x).$$

By 1(b), the latter statement is true (independent of x), so the initial statement is true.

3. Let $p(x)$ be the statement $x \in A$, $q(x)$ be the statement $x \in B$, and $r(x)$ be the statement $x \in C$. The strategy is to convert the set identity into a logical statement, namely

$$\forall x \in U : ((p(x) \vee q(x)) \wedge r(x)) \wedge ((q(x) \vee r(x)) \wedge p(x)) \Rightarrow r(x) \wedge p(x).$$

The demonstrate that the corresponding logical statement

$$((p \vee q) \wedge r) \wedge ((q \vee r) \wedge p) \Rightarrow r \wedge p$$

is a tautology using truth tables (or otherwise). It follows that the corresponding universal statement is true, so the set relationship is true, and we're done.

Alternatively, you could argue like this: suppose

$$x \in ((A \cup B) \cap C) \cap ((B \cup C) \cap A). \tag{1}$$

Then

$$x \in (A \cup B) \cap C,$$

from which it follows $x \in C$. Furthermore, it follows from (1) that

$$x \in (B \cup C) \cap A$$

from which it follows that $x \in A$. From $x \in C$ and $x \in A$ it follows that $x \in C \cap A$. We have shown that $x \in$ LHS implies $x \in RHS$, i.e., LHS $\subset$ RHS, as required.

The first method is more general: it always works when proving set identities. The second method is quicker, but requires a good intuitive sense of what is happening in the particular situation. You might try illustrating the previous identities/inclusions on Venn diagrams to help build up your intuition.

4. (a) Trying a few values of n we have $f(1) = 11$, $f(2) = 1$, $f(3) = 13$, $f(4) = 3$, $f(5) = 15$, $f(6) = 5$, and so on. From this we see that $f(n)$ is always odd (proof: if n is even, $f(n)$ is odd; if n is odd, $f(n)$ is odd; so $f(n)$ is always odd). It follows that f is not a surjection. (Alternatively, you can show very easily that $f(n) \neq 2$ for all $n \in \mathbb{N}$.)

Nor is f an injection. Analysis: Since the functions $n + 10$ and $n - 1$ are both injections, let's look for two values $n_1, n_2 \in \mathbb{N}$ such that $f(n_1) = 11 = f(n_2)$, where f uses a different formula for each value of n_i. Let's say $f(n_1) = n_1 + 10 = 11$ and $f(n_2) = n_2 - 1 = 11$, so $n_1 = 1$ and $n_2 = 12$. Checking, $f(1) = 11 = f(12)$, so f is not an injection.

(b) Let's try a few values of n. We have $g(1) = 1$; because $g(1)$ is odd we have $g(2) = 5g(1) + 1 = 6$; because $g(2)$ is even we have $g(3) = g(2)/2 = 3$; because $g(3)$ is odd we have $g(4) = 5g(3)+1 = 16$; $g(5) = g(4)/2 = 8$; $g(6) = g(5)/2 = 4$; $g(7) = g(6)/2 = 2$; $g(8) = g(7)/2 = 1$; and we're back where we started. Now $g(9) = 5g(8) + 1 = 6$, etc., so the whole set of values cycles. (We say that g is periodic with period 7 because $g(n + 7) = g(n)$ for any $n \in \mathbb{N}$. It follows that g is neither surjective nor injective; there is no value of n such that $g(n) = 5$, for example, and both $g(1) = 1$ and $g(8) = 1$.

5. f is a surjection because, for any $x \in S$, we have $f(y) = x$ where $y = f^5(x)$. Similarly, f is an injection because, if $f(x_1) = f(x_2)$, then $f^5(f(x_1)) = f^5(f(x_2))$, i.e., $f^6(x_1) = f^6(x_2)$, i.e., $x_1 = x_2$.

   Alternatively, f is a surjection because it has a right inverse g, namely $g = f^5$. Similarly, f is an injection because it has a left inverse, namely $g = f^5$. (It is not a coincidence that the left inverse and right inverse are the same. Why not?)

6. Pick $x \in S$. Then either $f^2(x) = x$ or $f^3(x) = x$, or both. In the first case, $f^6(x) = f^2(f^2(f^2(x))) = f^2(f^2(x)) = f^2(x) = x$. In the second case, $f^6(x) = f^3(f^3(x)) = f^3(x) = x$. So in either case, we have $f^6(x) = x$. Since there was no restriction on the $x \in S$ that we picked, we must have $f^6(x) = x$ true for all $x \in S$. It follows that the previous problem applies, so we can conclude that f is a bijection.

7. For all $x \in S$, we have $f^3(x) = f(f^2(x)) = x$. Substituting the value of $f^2(x)$ into the previous formula, we have $f(x) = x$ for all $x \in S$, which means that f is the identity function.

   Alternatively, $\forall x \in S : f^3(x) = x$ implies $\forall x \in S : f^2(f(x)) = x$ implies $\forall x \in S : f(x) = x$; here we make use of f^2 on the left instead of on the right.

8. We have $f^2(x) = f^3(f^2(x)) = f^5(x) = x$ for all $x \in S$, so we have both $\forall x \in S : f^2(x) = x$ and $\forall x \in S : f^3(x) = x$. Therefore the previous problem applies, and we can conclude f is the identity function.

9. This is one of the oldest documented problems in mathematics, from an ancient Hindu text. We work backwards. The only tricky part is reversing the functions "increased by three fourths of the result" and "diminished by one third of the result". The former can be written as $f(x) = x+(3/4)x = (7/4)x$, so the inverse is $f^{-1}(x) = (4/7)x$. Similarly, the latter is $g(x) = x - (1/3)x = (2/3)x$ so the inverse is $g^{-1}(x) = (3/2)x$. Working backwards one step at a time, we perform the following operations on the final answer: 2, multiplied by 10 to get 20, decreased by 8 to get 12, the square taken to get 144, augmented by 52 to get 196, then the square root taken to get 14, then multiplied by 3/2 to get 21, then multiplied by 7 to get 147, then multiplied by 4/7 to get 84, then divided by 3 to get 28. You should check that going forwards, starting with 28, you end at the final result of 2.

   The problem illustrates an important property of the inverse of a function, namely that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$; i.e., inverting the composition of functions is the same as the composition of the inverse functions *in the reverse order.*

10. Again, we work backwards from the end. The only tricky part is recognizing that taking a fifth from x results in $(4/5)x$, the inverse of which is $(5/4)x$. Suppose that after the final division each man gets k coconuts (in addition to the pile he has secretly stashed). Then before the division there were $5k$ coconuts in the pile. The last man to take a fifth in secret then returns his fifth, resulting in $(5/4)(5k)$. The monkey then returns his coconut, resulting in $(5/4)5k+1$ coconuts. After the fourth man reverses his nocturnal activities, there are

$$\frac{5}{4}\left(\frac{5}{4}5k + 1\right) + 1 = \frac{5^2}{4^2}5k + \frac{5}{4} + 1$$

coconuts. After the third man reverses his activities, there are

$$\frac{5}{4}\left(\frac{5^2}{4^2}5k + \frac{5}{4} + 1\right) + 1 = \frac{5^3}{4^3}5k + \frac{5^2}{4^2} + \frac{5}{4} + 1$$

coconuts. After the second man reverses his activities, there are

$$\frac{5}{4}\left(\frac{5^3}{4^3}5k + \frac{5^2}{4^2} + frac54 + 1\right) + 1 = \frac{5^4}{4^4}5k + \frac{5^3}{4^3} + \frac{5^2}{4^2} + \frac{5}{4} + 1$$

coconuts. Finally, after the first man reverses his activities, there are

$$\frac{5}{4}\left(\frac{5^4}{4^4}5k + \frac{5^3}{4^3} + \frac{5^2}{4^2} + \frac{5}{4} + 1\right) + 1 = \frac{5^5}{4^5}5k + \frac{5^4}{4^4} + \frac{5^3}{4^3} + \frac{5^2}{4^2} + \frac{5}{4} + 1$$

coconuts. Putting it all over a common denominator, we have

$$\frac{1}{4^5}\left(5^5(5k) + 5^4(4) + 5^3(4^2) + 5^2(4^3) + 5(4^4) + 4^5\right)$$

coconuts. The problem now is that most values of k result in fractional values for the final number of coconuts. The expression in the brackets above must be divisible by $4^5 = 2^{10} = 1024$ for the value k to work. We could proceed by trial and error, finding values of k that work, but if we are going to answer the bonus problem we should try a more general approach. We can simplify by noting that

$$5^5 + 5^4(4) + 5^3(4^2) + 5^2(4^3) + 5(4^4) + 4^5 = 5^6 - 4^6$$

(why?), so the initial number of coconuts can be written

$$\frac{1}{4^5}(5^6 - 4^6 + (5k-1)5^5) = \frac{1}{4^5}(5^5(5k+4) - 4^6)$$

which is not a fraction if an only if 1024 divides $5k + 4$. The smallest answer is when $5k + 4 = 1024$, $5k = 1020$, $k = 204$, and the initial number of coconuts is $k = 3121$. You should check that that answer actually works.

Other answers result when you have a value of n such that $5k + 4 = 1024n$ for natural numbers k and n. Some experimentation should convince you that every fifth n works, i.e., $n = 5m + 1$ where $m = 1, 2, 3, \ldots$, so $5k + 4 = 1024(5m + 1) \implies 5k = 1024(5m) + 1020 \implies k = 1024m + 204$ are the allowable values of k. I'll leave it up to you to find all the allowable values for the initial number of coconuts, and to check that they all work.

There are a at least a couple of other very slick ways of solving this problem. One is to use continued fractions, familiarity with which allowed Ramanujan to solve a similar problem the instant it was stated to him (see Robert Kanigel's biography of Ramanujan, *The Man Who Knew Infinity*, for more information), or by thinking in terms of negative coconuts: what if we start with something like -4 coconuts?