MATH221-001 200630 Problem Set 2

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Due: Friday, October 6, 2006

1. Verify that the following expressions are tautologies.

   (a) $\neg(p \wedge \neg q) \Leftrightarrow (\neg p \vee q)$

   (b) $p \wedge \neg(q \vee \neg r) \Rightarrow p \wedge \neg q$

2. Prove the following statements about sets by converting to logical identities.

   (a) $(A \cap B^c)^c = A^c \cup B$

   (b) $A \cap (B \cup C^c)^c \subset A \cap B^c$

   The notation A^c means the complement of the set A, i.e., the set $\{x \in U : x \notin A\}$ where U is the universal set for the given context. I.e., assuming that $x \in U$ for all x under consideration, we can write $x \in A^c \equiv \neg(x \in A)$.

3. Prove the following identity for sets:

   $$((A \cup B) \cap C) \cap ((B \cup C) \cap A) \subset C \cap A$$

   Hint: the set relation $\subset$ corresponds to the logical relation $\Rightarrow$.

4. Determine whether each of the following functions is an injection, a surjection, and/or a bijection.

   (a) $f : \mathbb{N} \to \mathbb{N}$ defined by

   $$f(n) = \begin{cases} n+10, & n \text{ odd} \\ n-1, & n \text{ even} \end{cases}$$

   (b) (Correction) $g : \mathbb{N} \to \mathbb{N}$ defined by

   $$g(n) = \begin{cases} 1, & n = 1 \\ g(n-1)/2, & n > 1,\ g(n-1) \text{ even} \\ 5g(n-1)+1, & n > 1,\ g(n-1) \text{ odd} \end{cases}$$

5. Show that if a function $f : S \to S$ has the property that

   $$\text{for all } x \in S,\ f^6(x) = x$$

   then f must be a bijection. (Here $f^6(x)$ means $f(f(f(f(f(f(x))))))$, i.e., f composed with itself six times altogether; similarly for $f^n(x)$ for any natural number n.)

6. Show that if the function $f : S \to S$ has the property that

   $$\text{for all } x \in S,\ f^2(x) = x \text{ or } f^3(x) = x$$

   then f must be a bijection. (Hint: use the previous problem.)

7. Show that if the function $f : S \to S$ has the property that

   $$\text{for all } x \in S,\ f^2(x) = x \text{ and } f^3(x) = x$$

   then f must be the identity function. (Recall that the identity function $id : S \to S$ is the function with the property $id(x) = x$ for all $x \in S$.)

8. Show that if the function $f : S \to S$ has the property that

$$\text{for all } x \in S,\ f^3(x) = x \text{ and } f^5(x) = x$$

then f must be the identity function.

9. What is the number that, multiplied by 3, then increased by three fourths of the result, then divided by 7, then diminished by one third of the result, then multiplied by itself, then diminished by 52, then the square root taken, then increased by 8, then divided by 10, gives the answer 2?

10. (The Monkey and the Coconuts.) After a plane crash, 5 men found themselves stranded on a desert island. The only food source on the island was coconuts, which were plentiful, and the only other inhabitant of the island was a monkey. The men spent their first day gathering coconuts, which they put into a large pile. They were so tired after gathering coconuts, they decided to leave the problem of dividing them until the next day. However, in the middle of the night, one of the men woke up and decided to take his share. He found that the pile of coconuts divided evenly into five with one coconut left over. He stashed his fifth, gave the leftover coconut to the monkey, left the rest of the coconuts in a pile for the others, and went back to sleep. However, a second man woke up afterwards, and also decided to do the same thing. Coincidentally, when he divided the remaining pile into five, there was again one left over for the monkey. He stashed his fifth. Each of the other three men went through the same process at some time during the night, each time with one leftover coconut for the monkey. In the morning when they woke up, none of them admitted to taking a portion already, so they divided the remaining pile into fifths, this time with none left over for the monkey. How many coconuts did they originally gather?

    (Bonus: find all solutions.)