MATH221-001 200630 Problem Set 3 Solutions DRAFT

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1. (a) By the definition of $(a+b)^2$,

$$(a+b)^2 = (a+b) \times (a+b).$$

By distributivity,

$$(a+b) \times (a+b) = ((a+b) \times a) + ((a+b) \times b).$$

By the other kind of distributivity,

$$((a+b) \times a) + ((a+b) \times b) = (a \times a + b \times a) + (a \times b + b \times b).$$

By associativity of addition applied twice,

$$(a \times a + b \times a) + (a \times b + b \times b) = (a \times a) + (b \times a + (a \times b + b \times b) = (a \times a) + ((b \times a + a \times b) + b \times b).$$

By commutativity of multiplication,

$$a \times a + ((b \times a + a \times b) + b \times b) = a \times a + ((a \times b + a \times b) + b \times b).$$

By the property of 1 and distributivity,

$$\begin{aligned}&(a \times a) + ((a \times b + a \times b) + b \times b)\\ &\qquad = a \times a + (((1 \times (a \times b))) + (1 \times (a \times b)) + b \times b)\\ &\qquad\qquad = (a \times a) + (((1+1) \times (a \times b)) + (b \times b)).\end{aligned}$$

By various conventions we can write

$$(a \times a) + (((1+1) \times (a \times b)) + (b \times b)) = (a^2) + ((2 \times (a \times b)) + (b^2)) = a^2 + 2ab + b^2.$$

All the care in the above is necessary in certain contexts, e.g., the study of abstract algebra where some rules (commutativity and even sometimes associativity) are dropped.

(b) Using the previous result with $2a$ put in place of a,

$$(2a+b)^2 = (2a)^2 + 2(2a)b + b^2.$$

By associativity of multiplication we have $2(2a) = (2 \times 2)a$; by the definition of 2 and 4 and distributivity we have $2 \times 2 = 2 \times (1+1) = (2 \times 1) + (2 \times 1) = 2 + 2 = 2 + (1+1) = (2+1) + 1 = 3 + 1 = 4$. Also, we have

$$(2a) \times (2a) = 2 \times (a \times 2a) = 2 \times (2a \times a) = (2 \times 2a) \times a = ((2 \times 2) \times a) \times a = (4 \times a) \times a$$

by associativity of multiplication, commutativity of multiplication, associativity, associativity, and the previous result that $2 \times 2 = 4$. Summarizing our results so far,

$$(2a+b)^2 = (4a) \times a + (4a) \times b + b^2 = 4a(a+b) + b^2$$

by distributivity.

2. (a) The last two digits should be your age.

   (b) The other digits should be the number you picked.

   (c) If you have had your birthday this year, the calculation is

$$n \to 2n \to 2n + 5 \to 50(2n + 5) \to 50(2n + 5) + 1756 - Y$$

   where Y is the year you were born. By distributivity,

$$50(2n + 5) + 1756 - Y = 100n + 250 + 1756 - Y = 100n + 2006 - Y.$$

   Multiplying n by 100 just adds two digits of 0 to the end of the number; since you have had your birthday, $2006 - Y$ is your age, which is then added to $100n$. Since the two last digits in $100n$ are 0, there is no carrying in the addition, so the last two digits remain equal to your age. If you haven't had your birthday yet, similar reasoning applies.

   (d) If your age is 100 or greater, the trick will not work because carrying may take place; in any case the last two digits can never be the age of someone 100 or over. Furthermore, if the year changes, the trick will not work as stated; the numbers 1756 and 1755 will have to be changed to current year minus 250 and current year minus 251 respectively.

3. Suppose first that $m = n$. Then by 1(a) we have

$$(m + n)^2 = m^2 + 2mn + n^2 = mn + 2mn + mn = 4mn,$$

where various conventions were applied. Now suppose that $m < n$. By the definition of $<$ we have $n = m + p$ for some natural number p. Then

$$(m + n)^2 = (m + m + p)^2 = (2m + p)^2.$$

Applying 1(b) we have

$$(m + n)^2 = 4m(m + p) + p^2 = 4mn + p^2.$$

Since p^2 is a natural number, we can write $(m + n)^2 > 4mn$ in this case. Finally, if $m > n$, we apply the above argument with m and n interchanged. In all cases, $(m + n)^2 \geq 4mn$.

That inequality also applies when m and n are any positive real numbers. Later in the course, we may discuss how to prove it in that case.

4. We prove the results using induction.

   (a) The base is $n = 1$, in which case we have $n(n + 1) = 1(2) = 2$ which is a multiple of 2. For the induction step, assume the induction hypothesis that $m(m+1)$ is a multiple of 2 for some $m \in \mathbb{N}$. Then $(m + 1)((m + 1) + 1) = (m + 1)(m + 1) + (m + 1) = m(m + 1) + (m + 1) + (m + 1) = m(m+1)+2(m+1)$. The latter expression is a multiple of 2 because $m(m+1)$ is a multiple of 2 by the induction hypothesis and $2(m + 1)$ clearly is a multiple of 2, and the sum of two multiples of 2 is a multiple of 2 by distributivity, which establishes the induction step. Since the base is true and the induction step holds, by induction we have $n(n+1)$ is a multiple of 2 for any natural number n.

   (b) The base is $n = 1$, in which case $n(n + 1)(n + 2) = 1(2)(3) = 6$ which is a multiple of 6. For the induction step, assume the induction hypothesis that $m(m+1)(m+2)$ is a multiple of 6 for some $m \in \mathbb{N}$. Then $(m+1)((m+1)+1)((m+1)+2) = (m+1)(m+2)(m+3) = (m+3)(m+1)(m+2) = m(m + 1)(m + 2) + 3(m + 1)(m + 2)$ by commutativity, distributivity, and various conventions. But $m(m + 1)(m + 2)$ is a multiple of 6 by the induction hypothesis, and $(m + 1)(m + 2)$ is a multiple of 2 by part (a) which implies $3(m + 1)(m + 2)$ is a multiple of 6. It follows that $(m + 1)((m + 1) + 1)((m + 1) + 2)$ is the sum of two multiples of 6 so is also a multiple of 6. That establishes the induction step, so the result is true for all $n \in \mathbb{N}$ by induction.

5. We could prove the result by induction, but given the results we have established already, there is a faster way. We have

$$n(n+1)(n+5) = n(n+1)(n+2) + 3(n+1)(n+2).$$

The first summand is a multiple of 6 by part (b) of the previous problem. The second summand is 3 times a multiple of 2 by part (a) of the previous problem; by associativity, that implies it is a multiple of 6. The sum of two multiples of 6 is a multiple of 6 by distributivity, so $n(n+1)(n+5)$ is always a multiple of 6.

6. The base is $n = 1$, in which case the left side of the proposed equality is $\sum_{i=1}^{1} i^2 = 1^2 = 1$ by the definition of $\sum$, and the right side of the proposed equality is $1(1+1)(2(1)+1)/6 = 1(2)(3)/6 = 1$. Therefore the proposed equality is true when $n = 1$ which establishes the base case.

   For the induction step, assume the induction hypothesis that

$$\sum_{i=1}^{m} i^2 = \frac{1}{6}m(m+1)(2m+1)$$

   for some $m \in \mathbb{N}$. Then for the induction step we have

$$\sum_{i=1}^{m+1} i^2 = \sum_{i=1}^{m} i^2 + (m+1)^2 = \frac{1}{6}m(m+1)(2m+1) + (m+1)^2 = \frac{1}{6}(m+1)\left[2m^2 + 7m + 6\right]$$

   by the definition of $\sum$, the induction hypothesis, and some algebra. We know what we're aiming for: we want to show that $2m^2 + 7m + 6 = ((m+1)+1)(2(m+1)+1)$, but it's easier to work backwards and show

$$((m+1)+1)(2(m+1)+1) = (m+2)(2m+3) = 2m^2 + 4m + 3m + 6 = 2m^2 + 7m + 6.$$

   That completes the induction step showing that

$$\sum_{i=1}^{m+1} i^2 = \frac{1}{6}(m+1)((m+1)+1)(2(m+1)+1)$$

   and the result follows for all $n \in \mathbb{N}$ by induction.

7. When $n = 1$ the proposed equality is $f_1 = f_3 - 1$ which is true because $f_1 = 1$ and $f_3 = 2$, which establishes the base. For the induction step, assume the induction hypothesis that

$$\sum_{i=1}^{m} f_i = f_{m+2} - 1$$

   for some $m \in \mathbb{N}$. But then

$$\sum_{i=1}^{m+1} f_i = \sum_{i=1}^{m} f_i + f_{m+1} = f_{m+2} - 1 + f_{m+1} = f_{m+1} + f_{m+2} - 1 = f_{m+3} - 1 = f_{(m+1)+2} - 1$$

   by the definition of sigma notation, the induction hypothesis, and the definition of the Fibonacci numbers respectively. That establishes the induction step, so the proposed equality holds for all $n \in \mathbb{N}$ by induction.

8. Note that the problem has been corrected from the erroneous version I originally posted. First, the base case $n = 1$: in that case, the purported inequality becomes $9 \times 2 \times 1 \geq 4 \times 3$ which is true. Now the induction step: assume the induction hypothesis $3^2 \times 2^m \times f_m \geq 2^2 \times 3^m$ for some m. Then

   $$3^2 \times 2^{m+1} \times f_{m+1} \geq 3^2 \times 2^{m+1} \times (f_m + f_{m-1})$$

   by the definition of the Fibonacci numbers.

   It looks like we need to have to change our strategy: we need to make use of the previous two results, so it looks like we're going to have to back track and use use strong induction. We also need to establish the result for two base cases, not just one. So we check $n = 2$: $3^2 \times 2^2 \times 1 \geq 2^2 \times 3^2$. which is true, so that establishes the first two cases as the base. To use strong induction, we also need to change our induction hypothesis: now we assume the strong induction hypothesis that $3^2 \times 2^m \times f_m \geq 2^2 \times 3^m$ for $m = 1, 2, \cdots, M$. The induction step is not to prove the result for $M + 1$. As before, we have

   $$3^2 \times 2^{M+1} \times f_{M+1} \geq 3^2 \times 2^{M+1} \times (f_M + f_{M-1}) = 2(3^2 2^M f_M) + 2^2(3^2 2^{M-1} f_{M-1}) \geq 2(2^2 3^M) + 4(2^2 3^{M-1})$$

   by the induction hypothesis. A little more algebra gives

   $$2(2^2 3^M) + 4(2^2 3^{M-1} = 6(2^2 3^{M-1} + 4(2^2 3^{M-1}) = 10(2^2 3^{M-1}) \geq 9(2^2 3^{M-1}) = 2^2 3^{M+1}.$$

   Altogether we have proved $3^2 \times 2^{M+1} \times f_{M+1} \geq 2^2 3^{M+1}$ which establishes the strong induction step. The result follows by induction.

   When we learn about rational numbers, the inequality established in this problem will tell us that $f_n \geq 3^{n-2}/2^{n-2} = (3/2)^{n-2}$ which gives us a lower bound on the size of the Fibonacci numbers.

9. (a) We argue by contradiction. If $p_1 \geq q_1$ then $p_1^2 \geq q_1^2$ so $2p_1^2 > q_1^2$ which contradicts $2p_1^2 = q_1^2$. So our assumption that $p_1 \geq q_1$ must be false, and we have proven that $p_1 < q_1$.

   (b) $q_1^2 = 2p_1^2$ so q_1^2 must be even.

   (c) Again, argue by contradiction. If q_1 were odd, then q_1^2 would be odd, but we know that q_1^2 is even. Therefore our assumption that q_1 is odd must be wrong, so q_1 must be even.

   (d) If q_1 is even we can write $q_1 = 2k$ for some k, so $q_1^2 = 4k^2$ is divisible by 4.

   (e) Since q_1 is divisible by 2 we can write $2(q_1/2)^2 = q_1^2/2 = p_1^2$, which gives us another pair of numbers $p_2 = q_1/2$, $q_2 = p_1$ satisfying the equation $2p^2 = q^2$. Furthermore we have $p_2^2 = q_1^2/4 < q_1^2/2 = p_1^2$ so $p_2 < p_1$, and $q_2 = p_1 < q_1$ by part (a).

   So, under the assumption that there is a solution in integers to the equation $2p^2 = q^2$, we have found another, smaller solution in integers. Repeating that process would lead to an infinite descending sequence of solutions, which, by Theorem 4.7, cannot exist. Therefore our assumption that there is a solution must be wrong. There is no solution in integers to $2p^2 = q^2$.

   When we learn about irrational numbers we will see that the above argument implies that $\sqrt{2}$ is irrational.

10. If S is bounded above, there is a natural number k such that $k \geq n$ for all $n \in S$. Consider the function $f : S \to \mathbb{N}$ given by $f(n) = k + 1 - n$. The set $f(S) = \{f(n) : n \in S\}$ is a set of natural numbers, so has a least element $m \in \mathbb{N}$. That means $m = f(s) = k + 1 - s$ for some $s \in S$ and $k + 1 - n \geq k + 1 - s$ for all $n \in S$. It follows that $s \geq n$ for all $n \in S$, so S has a greatest element.

    If the subtraction operation in the above bothers you, you can define a different function f that doesn't use subtraction. Your function just has to reverse the ordering of numbers in a given bounded subset of $\mathbb{N}$.

    Similar arguments apply to any subset of the integers. However, they do not apply to sets like the rational numbers; the set of rational numbers less than $\sqrt{2}$ is bounded above but does not have a greatest element.