MATH221-001 200630 Problem Set 3

Edward Doolittle

Due: Monday, October 23

1. Using Axioms 1–9 and any theorems we have already proven, show that, for any natural numbers a and b,

   (a) $(a+b)^2 = a^2 + 2ab + b^2$

   (b) $(2a+b)^2 = 4a(a+b) + b^2$

   (By convention, a^2 means $a \times a$, ab means $a \times b$, and multiplication is performed before addition.) State clearly at each step which axiom or theorem you are using.

2. Pick a natural number. Multiply it by 2. Add 5. Multiply by 50. If you have had your birthday already this year, add 1756; otherwise add 1755. Subtract the year in which you were born.

   (a) What is the significance of the last two digits of the resulting number?

   (b) What is the significance of the remaining digits?

   (c) How does the trick work?

   (d) Under what conditions will the above trick not work?

3. **The arithmetic-geometric mean inequality.** Using Axioms 1–10 and any theorems we have already proven, show that, for any natural numbers m and n, $4mn \leq (m+n)^2$. (Hint: Axiom 10 says $m < n$ or $m = n$ or $m > n$; consider each case, keeping in mind the definition of $a < b$. You may find the result of the first problem helpful.)

4. Using Axioms 1–11 and any theorems we have already proven, show that, for any natural number n,

   (a) $n(n+1)$ is a multiple of 2

   (b) $n(n+1)(n+2)$ is a multiple of 6

5. Show that $n(n+1)(n+5)$ is a multiple of 6 for any natural number n. (Hint: you may find the results of the previous problem useful.) There's no need to go into so much detail as in the previous problems; you can assume all the (correct) rules of algebra that you need.

6. Prove by mathematical induction that

$$\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$$

   for any natural number n.

7. Let f_i be the i^{th} Fibonacci number: $f_1 = f_2 = 1$ and $f_{i+2} = f_i + f_{i+1}$. Prove by mathematical induction that

$$\sum_{i=1}^{n} f_i = f_{n+2} - 1$$

   for all natural numbers n.

8. Again, let f_n be the n^{th} Fibonacci number. Use strong induction to prove that $3^2 \times 2^n \times f_n \geq 2^2 \times 3^n$ for all $n \in \mathbb{N}$.

9. **The method of infinite descent.** Suppose that there are natural numbers p_1 and q_1 such that $2p_1^2 = q_1^2$. Show that, in that case,

   (a) $p_1 < q_1$;

   (b) q_1^2 must be even (i.e., a multiple of 2);

   (c) so q_1 must be even;

   (d) so q_1^2 must be divisible by 4;

   (e) so there are numbers $p_2 = q_1/2$ and $q_2 = p_1$ such that $2p_2^2 = q_2^2$ but $p_2 < p_1$ and $q_2 < q_1$.

   Now let S be the set of numbers p such that $2p^2 = q^2$. Use Theorem 4.7 and the above results to show that S is the empty set, i.e., there is no solution in natural numbers to the equation $2p^2 = q^2$.

10. We say that a set S of natural numbers is **bounded above by** $k \in \mathbb{N}$ if $\forall n \in S : k \geq n$. We say that a set S of natural numbers is **bounded above** if it is bounded above by k for some $k \in \mathbb{N}$. Prove that, if a set of natural numbers is bounded above, it has a greatest element, i.e., an element $s \in S$ such that $\forall n \in S : n \leq s$.