MATH221-001 200630 Problem Set 5 Solutions DRAFT

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1. From now on, I'm going to use the symbol $\sim$ for a general equivalence relation rather than the symbol R. We need to show that the relation $\sim$ is reflexive, symmetric, and transitive, from which it will follow that it is an equivalence relation.

   Reflexive: $[a, b] \sim [a, b]$ is equivalent to $a+b = a+b$ which is obviously true. **Symmetric:** $[a, b] \sim [c, d]$ is equivalent to $a + d = b + c$ which is equivalent to $c + b = d + a$ which is equivalent to $[c, d] \sim [a, b]$. **Transitive:** $[a, b] \sim [c, d]$ and $[c, d] \sim [e, f]$ imply

   $$\begin{aligned} a + d &= b + c \\ c + f &= d + e. \end{aligned}$$

   Adding both of the above equations and then cancelling $c + d$ from both sides gives

   $$a + c + d + f = b + c + d + e \implies a + f = b + e \implies [a, b] \sim [e, f].$$

   We conclude that the relation $\sim$ is an equivalence relation.

2. Suppose we have a second representative of the equivalence class of $[a, b]$, call it $[a', b']$; similarly we have a second representative of the equivalence class of $[c, d]$, call it $[c', d']$. We must show that $[a', b'] - [c', d'] \sim [a, b] - [c, d]$, i.e., that the equivalence class of the difference of $[a, b]$ and $[c, d]$ does not depend on the choice of representatives of the equivalence classes of $[a, b]$ and $[c, d]$.

   In other words, our goal is to prove that

   $$[a' + d', b' + c'] \sim [a + d, b + c],$$

   i.e., $a' + d' + b + c = b' + c' + a + d$. We have the equations

   $$a + b' = b + a'c + d' \qquad\qquad = c' + d$$

   Flipping the first of the above equations around and adding we have

   $$a' + b + c + d' = a + b' + c' + d,$$

   which is what we need. The proof for multiplication is similar, so I won't go into detail.

3. (a) Applying the Euclidean Algorithm by dividing repeatedly, we have

   $$\begin{aligned} 2990 &= 2622 \times 1 + 328 \\ 2622 &= 368 \times 7 + 46 \\ 368 &= 46 \times 8 \end{aligned}$$

   so the greatest common divisor is 46.

   (b) Rewriting the above division equations gives

   $$\begin{aligned} 368 &= 2990 - 2622 \times 1 \\ 46 &= 2622 - 368 \times 7 \end{aligned}$$

from which we obtain

$$46 = 2622 - (2990 - 2622 \times 1) \times 7 = 2622 \times 8 + 2990 \times (-7).$$

So we have $d = 2990m + 2622n$ where $m = -7$ and $n = 8$.

(c) If we can solve the equation $2990a + 2622b = 0$ then we could add a to m and add b to n to get a new solution. But it's easy to solve that equation: we could have, for example, $a = 2622$ and $b = -2990$. So we have $d = 2990(-7+2622)+2622(8-2990) = 2990 \times 2615 + 2622(-2982)$. (We actually skipped over a few solutions; can you find solutions with m' somewhere between -7 and 2615? Can you find all solutions?)

(d) We just multiply both sides of the equation $46 = 2990(-7) + 2622(8)$ by 5 to obtain $230 = 2990(-35) + 2622(40)$.

(e) No, there is no solution to the given equation because 46 divides the left hand side but does not divide the right hand side, no matter what m_2 and n_2 are.

4. As with the previous problem, we begin by finding the greatest common divisor of 8 and 11 by the Euclidean Algorithm:

$$\begin{aligned} 11 &= 8 \times 1 + 3 \\ 8 &= 3 \times 2 + 2 \\ 3 &= 2 \times 1 + 1 \\ 2 &= 1 \times 2, \end{aligned}$$

so the greatest common divisor is 1. Now we re-write the above division equations (except for the last) to obtain expressions for the remainders:

$$\begin{aligned} 3 &= 11 \times 1 - 8 \times 1 \\ 2 &= 8 \times 1 - 3 \times 2 \\ 1 &= 3 \times 1 - 2 \times 1. \end{aligned}$$

Now we replace the second last number in the last equation above by its value according to the equation above it, so we have the two equations

$$\begin{aligned} 3 &= 11 \times 1 - 8 \times 1 \\ 1 &= 3 \times 1 - (8 \times 1 - 3 \times 2) \times 1 \end{aligned}$$

or in other words

$$\begin{aligned} 3 &= 11 \times 1 - 8 \times 1 \\ 1 &= 3 \times 3 - 8 \times 1. \end{aligned}$$

Now we eliminate the fourth last number in the last equation above using the first equation:

$$1 = (11 \times 1 - 8 \times 1) \times 3 - 8 \times 1 = 11 \times 3 - 8 \times 4$$

which is easy to check.

(a) In order to solve $11m + 8n = 777$, we multiply the equation $11 \times 3 - 8 \times 4 = 1$ by 777 to obtain

$$11 \times 2331 + 8 \times (-3108) = 777.$$

The problem with that equation is that we have a negative number of chicken dinners, which is impossible. So we start moving some of the value of lobster dinners over to chicken dinners. We can move 88 dollars worth at a time because that number is the least common multiple of 8 and

11. Another way to think of it is if I reduce the number of lobster dinners by 8, I increase the number of chicken dinners by 11.

We have $3108 = 11 \times 282 + 6$ so I need to repeat the process of increasing the number of chicken dinners by 11 283 times to get a positive number of chicken dinners. To maintain the same total value, I decrease the number of lobster dinners by 8 283 times as well. Altogether I get

$$\begin{aligned} 777 &= 11 \times (2331 - 8 \times 283) + 8 \times (-3108 + 11 \times 283) \\ &= 11 \times 67 + 8 \times 5 \end{aligned}$$

which you can check. We can again shuffle the value of 8 lobster dinners to 11 chicken dinners to obtain

$$777 = 11 \times (67 - 8) + 8 \times (5 + 11) = 11 \times 59 + 8 \times 16.$$

We can continue to repeat that process to get all the answers

$$\begin{aligned} 777 &= 11 \times 67 + 8 \times 5 \\ 777 &= 11 \times 59 + 8 \times 16 \\ 777 &= 11 \times 51 + 8 \times 27 \\ 777 &= 11 \times 43 + 8 \times 38 \\ 777 &= 11 \times 35 + 8 \times 49 \\ 777 &= 11 \times 27 + 8 \times 60 \\ 777 &= 11 \times 19 + 8 \times 71 \\ 777 &= 11 \times 11 + 8 \times 82 \\ 777 &= 11 \times 3 + 8 \times 93. \end{aligned}$$

If we continue the process any further, we get a negative number of lobster dinners, so we must stop. We conclude that we must have sold $3+8n$ lobster dinners, $n = 0, \ldots, 8$, and a corresponding number $93 - 11n$ chicken dinners, $n = 0, \ldots, 8$; there are 9 possible answers to this question, and we can figure out which one actually occurred without more information. (Can you prove that those are all the solutions?)

(b) In this case we have

$$96 = 11 \times (3 \times 96) + 8 \times (-4 \times 96) = 11 \times 288 + 8 \times (-384).$$

We have $384 = 11 \times 34 + 10$ so we must perform our shuffling process 35 times to get a positive number of chicken dinners. We have

$$96 = 11 \times (288 - 8 \times 35) + 8 \times (-384 + 11 \times 35) = 11 \times 8 + 8 \times 1.$$

We can shuffle one more time to obtain the complete set of solutions

$$\begin{aligned} 96 &= 11 \times 8 + 8 \times 1 \\ 96 &= 11 \times 0 + 8 \times 12 \end{aligned}$$

We still can't say for sure how many lobster dinners and how many chicken dinners we've sold, but if for example we know that we sold at least one of each we can figure out the exact number of each.

(c) As with the previous we have

$$69 = 11 \times (3 \times 69) + 8 \times (-4 \times 69) = 11 \times 207 + 8 \times (-276).$$

We have to shuffle the value of 8 lobster dinners over to 11 chicken dinners at least 26 times in this case, so we get

$$69 = 11 \times (207 - 8 \times 26) + 8 \times (-276 + 11 \times 26) = 11 \times -1 + 8 \times 10.$$

In this case we must have either the number of chicken dinners negative or the number of lobster dinners negative. I can conclude that it is impossible to get a value of \$69 by selling lobster dinners for \$11 and chicken dinners for \$8, so if the bill is \$69 there must be a mistake somewhere.

(Harder problem: find all the values that can't be achieved by adding up 11s and 8s, and prove that your answer is correct using strong induction.)

5. You can test each number $n < 200$ for primeness by dividing by every possible divisor $d < n$, but there are a lot of shortcuts you can use to speed up the work. For one thing, you need only divide by d up to the square root of n (why?). For another thing, you only need to consider the case d is prime (why?). You can use quick divisibility tests for $d = 2, 3, 5, 11$, leaving only 7 and 13 as the numbers to check by division. Probably the most efficient way of doing all these tests at once is the Sieve of Eratosthenes. We write all the numbers from 1 to 200 in an orderly fashion, as in Figure 1. Then we cross off every

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130
131	132	133	134	135	136	137	138	139	140
141	142	143	144	145	146	147	148	149	150
151	152	153	154	155	156	157	158	159	160
161	162	163	164	165	166	167	168	169	170
171	172	173	174	175	176	177	178	179	180
181	182	183	184	185	186	187	188	189	190
191	192	193	194	195	196	197	198	199	200

Table 1: Sieve of Eratosthenes I

even number except for 2 to obtain Table 2 The next number that appears uncrossed (3) must be a prime; we cross every third number after 3 to obtain Table 3. Note that the orderly arrangement of numbers allows us to find every third number easily. If a number is already crossed out (because it is a multiple of 2), we just leave it crossed. The next uncrossed number is 5, so we cross off every fifth number starting from 5 (but not 5) to obtain Table 4 The next uncrossed number is 7, so 7 is prime and we cross off every seventh number after 7 to obtain Table 5 The next uncrossed number is 11, so 11 is prime and we cross off every eleventh number from that point on to obtain Table 6. The next number uncrossed is 13, so 13 is prime and we cross off all other multiples of 13, which shouldn't be too hard to find because of the orderly arrangement of numbers in the table. See Table 7. We can stop at this point because 13 is the largest prime less than or equal to the square root of 200. That leaves the following set of primes up to 200:

$$\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, \\ 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, \\ 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, \\ 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199\}.$$

1	2	3	~~4~~	5	~~6~~	7	~~8~~	9	~~10~~
11	~~12~~	13	~~14~~	15	~~16~~	17	~~18~~	19	~~20~~
21	~~22~~	23	~~24~~	25	~~26~~	27	~~28~~	29	~~30~~
31	~~32~~	33	~~34~~	35	~~36~~	37	~~38~~	39	~~40~~
41	~~42~~	43	~~44~~	45	~~46~~	47	~~48~~	49	~~50~~
51	~~52~~	53	~~54~~	55	~~56~~	57	~~58~~	59	~~60~~
61	~~62~~	63	~~64~~	65	~~66~~	67	~~68~~	69	~~70~~
71	~~72~~	73	~~74~~	75	~~76~~	77	~~78~~	79	~~80~~
81	~~82~~	83	~~84~~	85	~~86~~	87	~~88~~	89	~~90~~
91	~~92~~	93	~~94~~	95	~~96~~	97	~~98~~	99	~~100~~
101	~~102~~	103	~~104~~	105	~~106~~	107	~~108~~	109	~~110~~
111	~~112~~	113	~~114~~	115	~~116~~	117	~~118~~	119	~~120~~
121	~~122~~	123	~~124~~	125	~~126~~	127	~~128~~	129	~~130~~
131	~~132~~	133	~~134~~	135	~~136~~	137	~~138~~	139	~~140~~
141	~~142~~	143	~~144~~	145	~~146~~	147	~~148~~	149	~~150~~
151	~~152~~	153	~~154~~	155	~~156~~	157	~~158~~	159	~~160~~
161	~~162~~	163	~~164~~	165	~~166~~	167	~~168~~	169	~~170~~
171	~~172~~	173	~~174~~	175	~~176~~	177	~~178~~	179	~~180~~
181	~~182~~	183	~~184~~	185	~~186~~	187	~~188~~	189	~~190~~
191	~~192~~	193	~~194~~	195	~~196~~	197	~~198~~	199	~~200~~

Table 2: Sieve of Eratosthenes II

1	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	25	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	35	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	49	~~50~~
~~51~~	~~52~~	53	~~54~~	55	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	65	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	77	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	85	~~86~~	~~87~~	~~88~~	89	~~90~~
91	~~92~~	~~93~~	~~94~~	95	~~96~~	97	~~98~~	~~99~~	~~100~~
101	~~102~~	103	~~104~~	~~105~~	~~106~~	107	~~108~~	109	~~110~~
~~111~~	~~112~~	113	~~114~~	115	~~116~~	~~117~~	~~118~~	119	~~120~~
121	~~122~~	~~123~~	~~124~~	125	~~126~~	127	~~128~~	~~129~~	~~130~~
131	~~132~~	133	~~134~~	~~135~~	~~136~~	137	~~138~~	139	~~140~~
~~141~~	~~142~~	143	~~144~~	145	~~146~~	~~147~~	~~148~~	149	~~150~~
151	~~152~~	~~153~~	~~154~~	155	~~156~~	157	~~158~~	~~159~~	~~160~~
161	~~162~~	163	~~164~~	~~165~~	~~166~~	167	~~168~~	169	~~170~~
~~171~~	~~172~~	173	~~174~~	175	~~176~~	~~177~~	~~178~~	179	~~180~~
181	~~182~~	~~183~~	~~184~~	185	~~186~~	187	~~188~~	~~189~~	~~190~~
191	~~192~~	193	~~194~~	~~195~~	~~196~~	197	~~198~~	199	~~200~~

Table 3: Sieve of Eratosthenes III

1	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	49	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	77	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
91	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~
101	~~102~~	103	~~104~~	~~105~~	~~106~~	107	~~108~~	109	~~110~~
~~111~~	~~112~~	113	~~114~~	~~115~~	~~116~~	~~117~~	~~118~~	119	~~120~~
121	~~122~~	~~123~~	~~124~~	~~125~~	~~126~~	127	~~128~~	~~129~~	~~130~~
131	~~132~~	133	~~134~~	~~135~~	~~136~~	137	~~138~~	139	~~140~~
~~141~~	~~142~~	143	~~144~~	~~145~~	~~146~~	~~147~~	~~148~~	149	~~150~~
151	~~152~~	~~153~~	~~154~~	~~155~~	~~156~~	157	~~158~~	~~159~~	~~160~~
161	~~162~~	163	~~164~~	~~165~~	~~166~~	167	~~168~~	169	~~170~~
~~171~~	~~172~~	173	~~174~~	~~175~~	~~176~~	~~177~~	~~178~~	179	~~180~~
181	~~182~~	~~183~~	~~184~~	~~185~~	~~186~~	187	~~188~~	~~189~~	~~190~~
191	~~192~~	193	~~194~~	~~195~~	~~196~~	197	~~198~~	199	~~200~~

Table 4: Sieve of Eratosthenes IV

1	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~
101	~~102~~	103	~~104~~	~~105~~	~~106~~	107	~~108~~	109	~~110~~
~~111~~	~~112~~	113	~~114~~	~~115~~	~~116~~	~~117~~	~~118~~	~~119~~	~~120~~
121	~~122~~	~~123~~	~~124~~	~~125~~	~~126~~	127	~~128~~	~~129~~	~~130~~
131	~~132~~	~~133~~	~~134~~	~~135~~	~~136~~	137	~~138~~	139	~~140~~
~~141~~	~~142~~	143	~~144~~	~~145~~	~~146~~	~~147~~	~~148~~	149	~~150~~
151	~~152~~	~~153~~	~~154~~	~~155~~	~~156~~	157	~~158~~	~~159~~	~~160~~
~~161~~	~~162~~	163	~~164~~	~~165~~	~~166~~	167	~~168~~	169	~~170~~
~~171~~	~~172~~	173	~~174~~	~~175~~	~~176~~	~~177~~	~~178~~	179	~~180~~
181	~~182~~	~~183~~	~~184~~	~~185~~	~~186~~	187	~~188~~	~~189~~	~~190~~
191	~~192~~	193	~~194~~	~~195~~	~~196~~	197	~~198~~	199	~~200~~

Table 5: Sieve of Eratosthenes V

1	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~
101	~~102~~	103	~~104~~	~~105~~	~~106~~	107	~~108~~	109	~~110~~
~~111~~	~~112~~	113	~~114~~	~~115~~	~~116~~	~~117~~	~~118~~	~~119~~	~~120~~
~~121~~	~~122~~	~~123~~	~~124~~	~~125~~	~~126~~	127	~~128~~	~~129~~	~~130~~
131	~~132~~	~~133~~	~~134~~	~~135~~	~~136~~	137	~~138~~	139	~~140~~
~~141~~	~~142~~	~~143~~	~~144~~	~~145~~	~~146~~	~~147~~	~~148~~	149	~~150~~
151	~~152~~	~~153~~	~~154~~	~~155~~	~~156~~	157	~~158~~	~~159~~	~~160~~
~~161~~	~~162~~	163	~~164~~	~~165~~	~~166~~	167	~~168~~	169	~~170~~
~~171~~	~~172~~	173	~~174~~	~~175~~	~~176~~	~~177~~	~~178~~	179	~~180~~
181	~~182~~	~~183~~	~~184~~	~~185~~	~~186~~	~~187~~	~~188~~	~~189~~	~~190~~
191	~~192~~	193	~~194~~	~~195~~	~~196~~	197	~~198~~	199	~~200~~

Table 6: Sieve of Eratosthenes VI

1	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~
101	~~102~~	103	~~104~~	~~105~~	~~106~~	107	~~108~~	109	~~110~~
~~111~~	~~112~~	113	~~114~~	~~115~~	~~116~~	~~117~~	~~118~~	~~119~~	~~120~~
~~121~~	~~122~~	~~123~~	~~124~~	~~125~~	~~126~~	127	~~128~~	~~129~~	~~130~~
131	~~132~~	~~133~~	~~134~~	~~135~~	~~136~~	137	~~138~~	139	~~140~~
~~141~~	~~142~~	~~143~~	~~144~~	~~145~~	~~146~~	~~147~~	~~148~~	149	~~150~~
151	~~152~~	~~153~~	~~154~~	~~155~~	~~156~~	157	~~158~~	~~159~~	~~160~~
~~161~~	~~162~~	163	~~164~~	~~165~~	~~166~~	167	~~168~~	~~169~~	~~170~~
~~171~~	~~172~~	173	~~174~~	~~175~~	~~176~~	~~177~~	~~178~~	179	~~180~~
181	~~182~~	~~183~~	~~184~~	~~185~~	~~186~~	~~187~~	~~188~~	~~189~~	~~190~~
191	~~192~~	193	~~194~~	~~195~~	~~196~~	197	~~198~~	199	~~200~~

Table 7: Sieve of Eratosthenes VII

6. The n numbers can be written

$$(n+1)!+2, (n+1)!+3, (n+1)!+4, \ldots, (n+1)!+(n+1).$$

   Since $(n+1)! = 1 \times 2 \times 3 \times \cdots \times n \times (n+1)$, we have that the first number in the above list is divisible by 2 (but is much larger than 2 so is not equal to 2), so is composite. The second number in the list is divisible by 3 but is not 3, etc.

7. **First solution.** Since d is a common divisor we can write $a = dm$ and $b = dn$. If m and n had a common factor, d would not be the greatest common divisor, so we know m and n have no common factor. Therefore $\text{GCD}(a/d, b/d) = \text{GCD}(m, n) = 1$.

   Second solution. We can write $ax + by = d$ for some x and y. Dividing by d (which goes into a and into b because it is a divisor of both) we have $(a/d)x + (b/d)y = 1$. It follows that the greatest common divisor of a/d and b/d divides 1 by a theorem we proved (for the most part) in chapter 4, so d must be 1.

8. We have $\text{GCD}(a+b, a-b) = \text{GCD}(a+b-(a-b), a-b) = \text{GCD}(2b, a-b)$. Multiplying the second argument by 2, we no longer have equality; however, since all the common divisors of $2b$ and $a-b$ are also divisors of $2b$ and $2(a-b)$ we can say

$$\text{GCD}(2b, a-b) | \text{GCD}(2b, 2(a-b)) = \text{GCD}(2b, 2a-2b+2b) = \text{GCD}(2b, 2a) = 2\,\text{GCD}(b, a) = 2.$$

   Altogether we have $\text{GCD}(a+b, a-b)|2$ which means that it must be 1 or 2.

9. Let p be the number of pennies, n the number of nickels, d the number of dimes, and q the number of quarters.

   (a) The total value of our collection of dimes and quarters is $10d + 25q$. That should be 1 dollar, i.e., 100 cents, so we have the equation $10d + 25q = 100$. We can divide through by the greatest common divisor of the terms to obtain $2d+5q = 20$. We can solve this by a method similar to that of the clambake question above, but it's easier just to look at the five possibilities $q = 0, 1, 2, 3, 4$; only $q = 0, 2, 4$ give solutions, so there are only three ways of making change for a dollar using dimes and quarters, namely 10 dimes, or 5 dimes and 2 quarters, or 4 quarters.

   (b) In this case the equation is $5n + 10d + 25q = 100$, or dividing by the greatest common divisor of the terms, $n + 2d + 5q = 20$. We could solve this in two steps, first looking for solutions to $x + 5q = 20$ and then solutions for $n + 2d = x$, but it's conceptually easier to look at the four possibilities for q: if $q = 0$ we need to solve $n + 2d = 20$ which has 11 solutions, $d = 0, \ldots, 10$ and $n = 20 - 2d$. If $q = 1$ we need to solve $n + 2d = 15$ which has 8 solutions, $d = 0, \ldots, 7$ and $n = 15 - 2d$. If $q = 2$ we have $n + 2d = 10$ which has 6 solutions; if $q = 3$ we have $n + 2d = 5$ which has 3 solutions, and if $q = 4$ we have 1 solution. Altogether we have $11+8+6+3+1 = 29$ solutions.

   (c) This is similar to the previous, so I won't go into detail. If you're looking for a challenge, you might try to solve the above problems by studying the coefficients of the infinite series

$$(1 + x^{10} + x^{20} + \cdots)(1 + x^{25} + x^{50} + x^{75} + \cdots)$$
$$(1 + x^5 + x^{10} + x^{15} + \cdots)(1 + x^{10} + x^{20} + \cdots)(1 + x^{25} + x^{50} + x^{75} + \cdots)$$
$$(1 + x + x^2 + x^3 + x^4 + x^5 + \cdots)(1 + x^5 + x^{10} + x^{15} + \cdots)(1 + x^{10} + x^{20} + \cdots)(1 + x^{25} + x^{50} + x^{75} + \cdots)$$

10. The equations are

$$p + d + q = 50p + 10d + 25q \qquad = 300.$$

   (The first equation counts the number of coins, the second counts their value.) Subtracting the first equation from the second we get

$$9d + 24q = 250.$$

But that is impossible since 3 divides the left hand side but does not divide the right hand side. So there is no solution to the system of equations.