UNIVERSITY OF REGINA
DEPARTMENT OF MATHEMATICS AND STATISTICS
MATH 221–001–002 200630

MATH 221 Sample Final Examination 200630

Time: 3 hours
Instructors:
Dr. Edward Doolittle (001)

Name: ______________________
Student #: ______________________
Section: ______________________

You have 3 hours to do each of the following questions. The test is worth a total of 100 marks. Please justify your conclusions and show all your work. No aids are permitted. Use the backs of the pages for rough work.

(marks)

(10) 1. (a) Find $d = \mathrm{GCD}(7205, 1705)$.

(b) Find integers x and y such that $d = 7205x + 1705y$.

(10) 2. Prove or disprove:

$$(A \cap B)^c \subset ((A \cap C) \cup (B \cap C))^c$$

for all sets A, B, C. (Hint: draw a Venn diagram and then either provide a counter-example or use an appropriate truth table.)

(10) 3. Consider the two sets $S = \{1^3, 2^3, 3^3, \ldots\}$ and $T = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$. Construct a bijection $f : S \to T$ by finding bijections $g : \mathbb{N} \to T$ and $h^{-1} : \mathbb{N} \to S$.

(10) 4. Show that $n^2 + 24n - 25$ is not prime for all $n \in \mathbb{Z}$. (Hint: induction is the wrong way to do this problem.)

(10) 5. Show that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$$

for all $n \in \mathbb{N}$.

(10) 6. Consider the relation $\sim$ on the plane $\mathbb{R}^2$ defined by

$$(x, y) \sim (z, w) \text{ if } x + y = z + w.$$

(a) Show that $(3, 2) \sim (1, 4)$ but $(8, -1) \not\sim (0, 9)$.

(b) Prove that $\sim$ is an equivalence relation.

(c) Determine the equivalence class of $(2, 3)$ and draw a picture of it in the plane $\mathbb{R}^2$.

(d) Prove that every element (x, y) of $\mathbb{R}^2$ is related by $\sim$ to one and only one element of the set $\{(a, 0) : a \in \mathbb{R}\}$.

(10) 7. Find the inverse of 63 modulo 128, and use your answer to find all solutions to the equation $63x = 119 \mod 128$.

(10) 8. Let a, b be relatively prime integers.

(a) Prove that $\mathrm{GCD}(a + b, a^2 + b^2) = 1$ or 2.

(b) Show that both of the above cases can actually occur.

(10) 9. We have four sets A, B, C, and D which are subsets of a universal set U. Suppose that

$$A \cup B \subset C^c$$
$$A \cup D = U$$
$$D = \varnothing$$

Is it necessarily true that $(C \cup D)^c = U$? (Hint: Consider the statements $p = x \in A$, $q = x \in B$, $r = x \in C$, $s = x \in D$. Construct a truth table and try to see whether the conditions can be true but the required conclusion can be false.)

(10) 10. (a) Show that if x is any odd integer, $x^2 \equiv 1 \bmod 8$, and that if x is any even integer, $x^2 \equiv 0$ or $4 \bmod 8$.

(b) Using the above fact, show that $x^2 + y^2 = 2006$ has no solutions.

(c) Using the above fact, find all solutions to $x^2 + y^2 = 1000$.