MATH221-001 200630 Sample Midterm Test 1 Solutions DRAFT

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1. (a) See Table 1 for the truth table for the given expressions. (Ignore the last column for now.) Since the columns under $\neg(p \wedge q)$ and $\neg q \vee \neg p$ are the same, the expressions are logically equivalent.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg q$	$\neg p$	$\neg q \vee \neg p$	$\neg q \vee \neg p \Leftrightarrow \neg(p \wedge q)$
F	F	F	T	T	T	T	T
F	T	F	T	F	T	T	T
T	F	F	T	T	F	T	T
T	T	T	F	F	F	F	T

Table 1: Truth table for $\neg q \vee \neg p \Leftrightarrow \neg(p \wedge q)$

(b) Now refer to the last column in Table 1. Since the last column is always true, the expression is a tautology.

(c) Let $p(x)$ be the statement $x \in A$ and $q(x)$ be the statement $x \in B$. Then the given set relationship can be written

$$\forall x \in U : x \in (B^c \cup A^c) \Leftrightarrow x \in (A \cap B)^c.$$

Expanding using the definitions of set union, complement, and intersection, the statement is equivalent to

$$\forall x \in U : \neg p(x) \vee \neg q(x) \Leftrightarrow \neg(p(x) \wedge q(x));$$

however, the statement $\neg p(x) \vee \neg q(x) \Leftrightarrow \neg(p(x) \wedge q(x))$ is always true by the logical analysis in part (b), so the given set relationship is true.

2. Let us first work out the effect of $f \circ g$ on an element of $\mathbb{N}$.

$$(f \circ g)(n) = f(g(n)) = f(5n + 1).$$

At this point we should note that we can write $f(u) = 3u + 1$ for any u (n in the definition of f is a dummy variable and can be replaced by anything as long as the replacement is done consistently); letting $u = 5n + 1$ we have

$$f(5n + 1) = 3(5n + 1) + 1 = 15n + 3 + 1 = 15n + 4.$$

On the other hand, we have

$$(g \circ f)(n) = g(f(n)) = g(3n + 1) = 5(3n + 1) + 1 = 15n + 5 + 1 = 15n + 6.$$

The two functions $f \circ g$ and $g \circ f$ differ. If we want to be really pedantic about it, we should find an example of a specific value of n where they differ, e.g., $(f \circ g)(1) \neq (g \circ f)(1)$.

Alternatively, the calculations above can be done by evaluating the outer function first, e.g.,

$$(f \circ g)(n) = f(g(n)) = 3g(n) + 1 = 3(5n + 1) + 1 = 15n + 3 + 1 = 15n + 4.$$

Either way, you should get the same result for $(f \circ g)(n)$.

3. To get a feel for this, we should first evaluate $f(n)$ for a few values of n: $f(1) = 3$, $f(2) = 7$, $f(3) = 13$, $f(4) = 21$, etc. It appears that the values are increasing, and not all values are hit. Since f is increasing, it must be an injection: each value is greater than all previous values, so can't be equal to any previous value. Since there are values that aren't hit, f cannot be a surjection; $f(n) \geq 3$ for all n so $f(n)$ can never equal 1, for example.

   The above is quite acceptable to me, but you may feel more comfortable if the arguments are formalized somewhat. For example, we could argue like this: If $f(n_1) = f(n_2)$, there are three cases: $n_1 > n_2$, $n_1 < n_2$, and $n_1 = n_2$. In the first case, $n_1^2 > n_2^2$, $n_1 > n_2$, and $1 = 1$, so adding the inequalities we have $n_1^2 + n_1 + 1 > n_2^2 + n_2 + 1$, contradicting our assumption that $f(n_1) = f(n_2)$. The second case, $n_1 < n_2$, leads to the same sort of conclusion by similar reasoning. Therefore the only possible case is $n_1 = n_2$, so the function is an injection.

   To prove that it is a surjection, we just have to show that the equation $f(n) = 1$ has no solution in $\mathbb{N}$. The equation becomes $n^2 + n + 1 = 1$, i.e., $n^2 + n = 0$, which factors to $n(n + 1) = 0$ with the solutions $n = 0$ and $n = -1$, neither of which is in $\mathbb{N}$.

4. (a) Since g is not an injection there are s_1 and s_2 such that $g(s_1) = g(s_2)$ but $s_1 \neq s_2$. Then $(f \circ g)(s_1) = f(g(s_1)) = f(g(s_2)) = (f \circ g)(s_2)$ but $s_1 \neq s_2$ so $f \circ g$ is not an injection.

   (b) Consider as an example the functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 2n$ and

   $$g(n) = \begin{cases} n, & n \text{ odd} \\ n - 1, & n \text{ even} \end{cases}$$

   It is easy to verify that f is an injection while g is not. However, the composition is $(g \circ f)(n) = g(f(n)) = g(2n) = 2n - 1$ (because $f(n) = 2n$ is even); note that the composition is an injection.

5. This problem is rather hard. Let's look at a few values to get a feeling for f: $f(1) = 1^2 + 3 = 4$, $f(2) = 6(2)+2 = 14$, $f(3) = 3^2+3 = 12$, $f(4) = 6(4)+2 = 26$, $f(5) = 5^2+3 = 28$, $f(6) = 6(6)+2 = 38$. It appears that the list is increasing, but trying $f(7) = 7^2 + 3 = 52$ and $f(8) = 6(8) + 2 = 50$ breaks that pattern. It's easy enough to show that $f(n_1) \neq f(n_2)$ if both n_1, n_2 are odd because $n^2 + 3$ is increasing; similarly, if both n_1, n_2 are even, the $f(n_1) \neq f(n_2)$ because $6n + 2$ is increasing. The problem comes if one of n_1, n_2 is odd and the other is even.

   A little bit of thought or divine inspiration may lead you to notice that all of the values of f are even numbers; dividing by 2 we get $f(1)/2 = 2$, $f(3)/2 = 6$, $f(5)/2 = 14$, $f(7)/2 = 26$, all even numbers, but $f(2)/2 = 7$, $f(4)/2 = 13$, $f(6)/2 = 19$, $f(8)/2 = 25$, all odd numbers. If that pattern persists, we can see that $f(n)$ is divisible by 4 if n is odd, but $f(n)$ is an even number not divisible by 4 if n is even. We can verify that with a little bit of algebra: $n^2 + 3 = (2k + 1)^2 + 3 = 4k^2 + 4k + 4$ if n is an odd number of the form $2k + 1$; on the other hand, if n is an even number of the form $2k$, then $6(2k) + 2 = 12k + 2 = 4(3k) + 2$ which is not a multiple of 4 for any k. So we can never have $f(n_1) = f(n_2)$ when one of n_1, n_2 is odd and the other even.

6. This is another hard problem. For the sake of simpler notation, let $S = \{1, 2, 3, \ldots, s\}$ where s is the size of S. Then we can write $f^{k_1}(1) = 1$, $f^{k_2}(2) = 2$, $\ldots$, $f^{k_s}(s) = s$, i.e., for each value $n \in S$, we have a corresponding k_n.

   It's straightforward to show that f is a surjection. Given any $n \in S$, we have $f(y) = n$ where $y = f^{k_n-1}(n)$. To show that f is an injection is somewhat harder. Suppose $f(m) = f(n)$. Then $f^{k_m}(m) = m$ and $f^{k_n}(n) = n$. The problem is that f may be raised to different powers on both sides. Suppose for example that $f(m) = f(n)$ but $f^2(m) = m$ and $f^3(n) = n$. We can find a way around this using a trick from the problem sets: look at f^6. In detail, we have $f^5(f(m)) = f^5(f(n))$, so $f^6(m) = f^6(n)$, so $f^2(f^2(f^2(m))) = f^3(f^3(n))$, so $m = n$. The general argument works the same, but instead of 6 we use $k_m \times k_n$.

   As an alternative solution, multiply those k_s all together to get $k^* = k_1 \times k_2 \times \cdots \times k_s$, then $f^{k^*}(1) = f^{k_1}(f^{k_1}(\cdots f^{k_1}(1)\cdots)) = 1$, where the number of f^{k_1} in the above is k^*/k_1. Similarly $f^{k^*}(2) = 2$, etc., so f^{k^*} is the identity function. Then f is a surjection because $f(f^{k^*-1}(n)) = n$ for any $n \in S$, and f is an injection because $f(n_1) = f(n_2)$ implies $f^{k^*-1}(f(n_1)) = f^{k^*-1}(f(n_2))$ which implies $f^{k^*}(n_1) = f^{k^*}(n_2)$ which implies $n_1 = n_2$.

The second solution only works when S is finite. When S is infinite, the first solution still works.