MATH221-001 200630 Sample Midterm Test 1

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You have 50 minutes to do the following test. The test is worth 50 marks; you should try to earn one mark per minute. No aids (calculators, notes, etc.) are permitted. You can use the backs of the pages for rough work.

1. (a) (5 marks) Show that the logical expressions $\neg(p \wedge q)$ and $\neg q \vee \neg p$ are equivalent.
   (b) (5 marks) Show that the logical expression $\neg q \vee \neg p \Leftrightarrow \neg(p \wedge q)$ is a tautology.
   (c) (5 marks) Show that, for any two sets A and B, $B^c \cup A^c = (A \cap B)^c$.
2. (5 marks) Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = 3n + 1$, and let $g : \mathbb{N} \to \mathbb{N}$ be defined by $g(n) = 5n + 1$. Show that the two compositions $f \circ g$ and $g \circ f$ are not equal.
3. (10 marks) Determine whether the function $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) = n^2 + n + 1$ is a surjection, injection, and/or a bijection.
4. Suppose $f : S \to S$ is an injection, but $g : S \to S$ is not.
   (a) (5 marks) Prove that $f \circ g$ is never an injection.
   (b) (5 marks) Give an example of such f and g where $g \circ f$ is an injection. (Hint: S must be infinite; take $S = \mathbb{N}$.)
5. (5 marks) Is the function $f : \mathbb{N} \to \mathbb{N}$ defined by
$$f(n) = \begin{cases} n^2 + 3, & n \text{ odd} \\ 6n + 2, & n \text{ even} \end{cases}$$
   an injection?
6. (5 marks) Suppose you have a function $f : S \to S$ with the property that, given any $x \in S$, there is a $k \in \mathbb{N}$ such that $f^k(x) = x$. Show that, if S is finite, f must be a bijection. Bonus: what can you say if S is infinite?