

# MATH221-001 200630 Sample Midterm Test 2 Solutions DRAFT

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1. Proof by induction. The base is  $n = 1$ , in which case the LHS of the proposed equality is  $4(1) + 3 = 7$  and the RHS is  $2(1)^2 + 5(1) = 7$ , so the proposed equality is true for  $n = 1$ . To prove the induction step, assume the induction hypothesis that the proposed equality is true for  $n = m$  for some  $m \in \mathbb{N}$ . Then

$$\sum_{r=1}^{m+1} (4r + 3) = \sum_{r=1}^m (4r + 3) + 4(m + 1) + 3 = 2m^2 + 5m + 4m + 7 = 2m^2 + 9m + 7 = 2(m + 1)^2 + 5(m + 1)$$

which shows that the proposed inequality is true for  $n = m + 1$ . That establishes the induction step. The equality then holds for any  $n \in \mathbb{N}$  by the principle of induction.

2. (a) Let  $a$  be the least number in the set, then the elements of the set are  $a, a + 1, a + 2, \dots, a + 19$ . Let the pigeonholes be  $\{a, a + 1\}, \{a + 2, a + 3\}, \{a + 4, a + 5\}, \dots, \{a + 18, a + 19\}$ . There are 10 pigeonholes (count them), so if I choose 11 numbers, two must come from the same pigeonhole hence must differ by 1.  
(b) Choose  $a, a + 2, a + 4, \dots, a + 18$ . That is 10 numbers (count them), but no two of those numbers differ by 1.
3. Multiply the first formula by 3 and the second by 2 to obtain

$$6 \sum_{r=1}^n r = 3n(n + 1)$$
$$6 \sum_{r=1}^n (r^2 + r) = 2n(n + 1)(n + 2).$$

Now subtract the former from the latter to obtain

$$6 \left( \sum_{r=1}^n (r^2 + r) - \sum_{r=1}^n r \right) = 2n(n + 1)(n + 2) - 3n(n + 1)$$
$$6 \sum_{r=1}^n r^2 = n(n + 1)(2n + 4 - 3) = n(n + 1)(2n + 1).$$

4. Since each  $x_n$  is defined in terms of the previous two values, we will have to establish the first two cases and then use strong induction to fill in the rest. When  $n = 1$ ,  $x_n = 1$  by definition and  $4^n - 3^n = 4 - 3 = 1$  so the two sides agree. When  $n = 2$ ,  $x_n = 7$  and  $4^n - 3^n = 16 - 9 = 7$  so the two sides agree again.

For the induction step, we assume the induction hypothesis that the result is true for every value of  $n$  for  $n \leq M$  where  $M$  is  $\geq 2$ . To establish the induction step we want to prove that it is true for  $n = M + 1$ . To that end, for  $M \geq 2$  we write

$$x_{M+1} = 7x_M - 12x_{M-1} = 7(4^M - 3^M) - 12(4^{M-1} - 3^{M-1})$$

where the last equality holds by the induction hypothesis. Doing a little algebra,

$$x_{M+1} = 7 \times 4^M - 7 \times 3^M - 3(4 \times 4^{M-1}) + 4 \times (3 \times 3^{M-1}) = 4 \times 4^M - 3 \times 3^M = 4^{M+1} - 3^{M+1},$$

as required to establish the induction step. The result follows for all  $n \in \mathbb{N}$  by strong induction.

5. Let  $T = \{n(n+1)/2 : n \in \mathbb{N}\}$  be the set of triangular numbers. Let  $f : \mathbb{N} \rightarrow T$  be the function given by  $f(n) = n(n+1)/2$ . Then  $f$  is a surjection because  $T$  is defined to be all the numbers that are images of  $f$ , and  $f$  is an injection because it is increasing. (Fill in the details.) It follows that  $f$  is a bijection so has an inverse  $g = f^{-1}$ . Then we have a bijection  $g : T \rightarrow \mathbb{N}$ , which shows that  $T$  is countable.

6. This is a slight variation on the example of section 6.4 of the textbook. Everyone is acquainted with themselves, so the number of possible acquaintances a person may have is at least 1 and at most  $n$ , which by itself isn't good enough to prove the result by the pigeonhole principle. We must further consider two cases: if there is a person in the room who is acquainted with everyone, or if there is no such person in the room.

If there is a person who is acquainted with everyone, then no one can be acquainted with just one person so the number of acquaintances that anyone in the room may have is a number from 2 to  $n$ , just  $n - 1$  different values for  $n$  people. By the pigeonhole principle, some two people must have the same number of acquaintances.

On the other hand if there is no person in the room who is acquainted with everyone, then the number of acquaintances that a person in the room may have is between 1 and  $n - 1$ , again just  $n - 1$  different values for  $n$  people, so again by the pigeonhole principle there must be two people with the same number of acquaintances.

7. This is just a variation on problem 10 from Problem Set 4, which is the special case when  $n = 50$ . The pigeonholes are the set of all numbers which map to 1, the set of all numbers that map to 3,  $\dots$ , the set of all numbers that map to  $2n - 1$ .

(a) There are  $n$  pigeonholes, so if  $|X| \geq n + 1$  to elements of  $X$  get mapped to the same value by  $f$ , i.e.,  $f$  is not an injection.

(b) By the previous part there are two elements  $x_1, x_2$  of  $X$  such that  $f(x_1) = f(x_2) = y$ . Then  $x_1$  and  $x_2$  can have no greater odd factors so  $x_1 = 2^{n_1}y$  and  $x_2 = 2^{n_2}y$  some powers of  $n_1, n_2$  (here  $n_1$  or  $n_2$  could be 0, which shouldn't cause you too much grief; the result could be reformulated without 0 but it would just sound more complicated than necessary). If  $n_1 < n_2$  then  $x_1$  divides  $x_2$ ; if  $n_1 > n_2$  then  $x_2$  divides  $x_1$ .

8. The algebra is straightforward:  $p_1^2 = 2q_1^2$  implies that

$$2q_2^2 = 2(p_1 - q_1)^2 = 2p_1^2 - 4p_1q_1 + 2q_1^2 = p_1^2 - 4p_1q_1 + p_1^2 + 2q_1^2 = p_1^2 - 4p_1q_1 + 4q_1^2 = (p_1 - 2q_1)^2 = p_2^2.$$

We also need to check that  $q_2 = p_1 - q_1$  and  $p_2 = 2q_1 - p_1$  are natural numbers. Clearly they are integers since  $p_1$  and  $q_1$  are integers, so we just need to check whether  $p_2 > 0$  and  $q_2 > 0$ . We argue by contradiction (i.e., prove the contrapositive statement). If  $p_1 \leq q_1$  then  $p_1^2 \leq q_1^2 < 2q_1^2$  which contradicts our assumption that  $x^2 = 2y^2$ , so we must have  $p_1 > q_1$ , so  $q_2 > 0$ . Similarly, if  $p_1 \geq 2q_1$  then multiplying through by  $p_1 > q_1$  (which we know is true from the previous argument) we have  $p_1^2 > 2q_1^2$ . That again contradicts our assumption that  $p_1^2 = 2q_1^2$ , so again we must have  $p_1 < 2q_1$ , i.e.,  $q_2 > 0$ .

In order to show that  $S$  has no least element, we argue as follows. Suppose  $p_1 \in S$ . Then there is a natural number  $q_1$  such that  $p_1^2 = 2q_1^2$ . By the above, we can construct another pair of natural numbers  $(p_2, q_2)$  with the property that  $p_2^2 = 2q_2^2$ . That means that we also have  $p_2 \in S$ . We can repeat the above reasoning to construct another pair  $(p_3, q_3) = (2q_2 - p_2, p_2 - q_2)$  with the property that  $p_3^2 = 2q_3^2$  and so on for  $(p_4, q_4), (p_5, q_5), \dots$ . Furthermore, we have  $p_1 > p_2 > p_3 > \dots$  by the above reasoning. It follows that our set  $S$  has no least member. By theorem 4.7 in the textbook,  $S$  must be empty; there is no integer  $p_1$  such that  $p_1^2 = 2q_1^2$  for some integer  $q_1$ .

This gives an alternative argument to show that  $\sqrt{2}$  is irrational.