MATH221-001 200630 Sample Midterm Test 2

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You have 50 minutes to do the following test. The test is worth 50 marks; you should try to earn one mark per minute. No aids (calculators, notes, etc.) are permitted. You can use the backs of the pages for rough work.

The last two problems are harder than the others. You should attempt them only after you have finished the other problems.

1. (5 marks) Show that
$$\sum_{r=1}^{n}(4r+3) = 2n^2+5n$$
for all $n \in \mathbb{N}$.

2. (a) (5 marks) Suppose I have a set of 20 consecutive numbers, for example, the set might be $\{14, 15, 16, \ldots, 33\}$. Show that, if I choose 11 numbers from such a set, I must have two numbers that differ by 1.

   (b) (5 marks) Show that I can choose 10 numbers from such a set and not have two numbers which differ by 1.

3. (5 marks) Recall that we proved the formulas
$$2\sum_{r=1}^{n} r = n(n+1)$$
$$3\sum_{r=1}^{n} r(r+1) = n(n+1)(n+2)$$
Use those formulas to find a formula for $\sum_{r=1}^{n} r^2$. (Do not prove your formula by induction!)

4. (5 marks) Define the numbers x_n recursively by $x_1 = 1$, $x_2 = 7$, and $x_{n+2} = 7x_{n+1} - 12x_n$ for $n = 1, 2, \ldots$. Use strong induction to prove that
$$x_n = 4^n - 3^n$$
for all $n \in \mathbb{N}$.

5. (5 marks) Show that the set of triangular numbers $\{n(n+1)/2 : n \in \mathbb{N}\}$ is countable.

6. (5 marks) There are $n \in \mathbb{N}$ people in a room. Given any two people, either they are acquaintances or they are not acquaintances. Show that there are two people in the room who have the same number of acquaintances in the room.

7. Let X be a subset of $\{1, 2, \ldots, 2n\}$, and let Y be the set of odd numbers $\{1, 3, \ldots, 2n-1\}$. Define the function $f : X \to Y$ by $f(x) =$ the greatest member of Y that exactly divides x. For example, if $n = 5$ then

$$f(1) = f(2) = f(4) = f(8) = 1 \quad f(3) = f(6) = 3 \quad f(5) = f(10) = 5 \quad f(7) = 7 \quad f(9) = 9$$

   (a) (5 marks) Show that if $|X| \geq n+1$ then f cannot be an injection.

   (b) (5 marks) Show that if $|X| \geq n+1$ then there must be two numbers $x_1, x_2 \in X$ such that x_1 divides x_2.

8. (5 marks) Show that if p_1 and q_1 are natural numbers satisfying $p^2 = 2q^2$, then it is also true that $p_2 = 2q_1 - p_1$ and $q_2 = p_1 - q_1$ are solutions to $p^2 = 2q^2$. Now let $S = \{p \in \mathbb{N} : p^2 = 2q^2\}$. Show that S must be empty.