MATH281 200610 Midterm Test 2 Solutions DRAFT

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1. The associated homogeneous equation is $y''' + y'' = 0$ with auxiliary equation $m^3 + m^2 = 0$. The auxiliary equation factors as $m^2(m+1) = 0$ so has roots $m = 0, 0, -1$. The general solution to the homogeneous equation is $y_h = c_1 + c_2 x + c_3 e^{-x}$. To find a particular solution, the best method is undetermined coefficients. We search for a solution of the form $y_p = Ax^2 + Bx + C$. Since the last two terms of the proposed particular solution are in the kernel of the differential operator, we can drop them and search for a solution of the form $y_p = Ax^2$. Differentiating, $y_p' = 2Ax$, $y_p'' = 2A$, $y_p''' = 0$, so the condition becomes $0 + 2A = 6$, i.e., $A = 3$. So the general solution to the differential equation is

$$y = c_1 + c_2 x + c_3 e^{-x} + 3x^2.$$

You should check the solution.

2. The associated homogeneous equation is $y'' - 2y' + 2y = 0$ with auxiliary equation $m^2 - 2m + 2 = 0$. The roots of the auxiliary equation are

$$m = \frac{2 \pm \sqrt{4 - 4(2)}}{2} = 1 \pm i.$$

The general solution to the homogeneous equation is

$$y_h = c_1 e^x \cos x + c_2 e^x \sin x.$$

To find a particular solution, undetermined coefficients won't work, so we use variation of parameters instead. The Wronskian of the fundamental system of solutions $e^x \cos x$, $e^x \sin x$ is

$$W = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x \cos x - e^x \sin x & e^x \sin x + e^x \cos x \end{vmatrix} = e^{2x} \cos x \sin x + e^{2x} \cos^2 x - e^{2x} \cos x \sin x + e^{2x} \sin^2 x = e^{2x}.$$

The other determinants are

$$W_1 = \begin{vmatrix} 0 & e^x \sin x \\ e^x \tan x & e^x \sin x + e^x \cos x \end{vmatrix} = -e^{2x} \sin x \tan x$$

$$W_2 = \begin{vmatrix} e^x \cos x & 0 \\ e^x \cos x - e^x \sin x & e^x \tan x \end{vmatrix} = e^{2x} \sin x.$$

It follows that

$$c_1(x) = \int \frac{W_1}{W}\, dx = -\int \frac{e^{2x} \sin x \tan x}{e^{2x}}\, dx = -\int \sec x - \cos x \, dx = -\ln|\sec x + \tan x| + \sin x$$

$$c_2(x) = \int \frac{W_2}{W}\, dx = \int \frac{e^{2x} \sin x}{e^{2x}}\, dx = \int \sin x \, dx = -\cos x$$

so a particular solution to the equation is

$$y_p = (-\ln|\sec x + \tan x| + \sin x) e^x \cos x - \cos x e^x \sin x = -e^x \ln|\sec x + \tan x|$$

and a general solution is

$$y = c_1 e^x \cos x + c_2 e^x \sin x - e^x \ln|\sec x + \tan x|.$$

You should check the solution.

3. We have

$$
\begin{aligned}
y_1 &= \ln x \\
y_1' &= \frac{1}{x} \\
y_1'' &= -\frac{1}{x^2}
\end{aligned}
$$

so

$$
x^2 y_1'' + x y_1' = -x^2 \frac{1}{x^2} + x \frac{1}{x} = -1 + 1 = 0
$$

and $y_1 = \ln x$ satisfies the differential equation. Similarly

$$
\begin{aligned}
y_2 &= \ln x^2 \\
y_2' &= \frac{1}{x^2} \cdot 2x = \frac{2}{x} \\
y_2'' &= -\frac{2}{x^2}
\end{aligned}
$$

so

$$
x^2 y_2'' + x y_2' = -x^2 \frac{2}{x^2} + x \frac{2}{x} = -2 + 2 = 0,
$$

and y_2 also satisfies the differential equation. To see whether the two given functions forms a fundamental system of solutions, find the Wronskian:

$$
W = \begin{vmatrix} \ln x & \ln(x^2) \\ \frac{1}{x} & \frac{2}{x} \end{vmatrix} = \frac{2 \ln x}{x} - \frac{\ln(x^2)}{x} = \frac{\ln x^2 - \ln x^2}{x} = 0
$$

by the laws of logarithms, so the two functions are linearly dependent (in fact, $y_2 = 2y_1$) and so do not form a fundamental system of solutions.

We need a second solution. There are numerous ways to find a second solution. I will suggest three, in increasing order of difficulty.

Guessing. It is easy to guess a second solution in this case because the differential equation has no term of the form $a_2(x)y$. Therefore the constant function $y = 1$ is a solution to the homogeneous equation. The Wronskian $W(\ln x, 1) = -1/x \neq 0$, so $\ln x$, 1 is a fundamental system of solutions.

Reduction to a first order equation. We could use reduction of order (see below), but the whole point of reduction of order is to put the equation into a form in which no term of the form $a_2(x)y$ appears, and the equation is already in that form! Continuing, let $v = y'$. Then the equation becomes the first order linear homogeneous (separable) equation

$$
x^2 v' + xv = 0 \implies \frac{dv}{v} = -\frac{dx}{x}
$$

with general solution

$$
\ln v = -\ln x + C \implies v = \frac{c_1}{x} \implies y = \int \frac{c_1}{x}\, dx = c_1 \ln x + c_2
$$

so again we see that a fundamental system of solutions is $\ln x$, 1.

Variation of parameters. Variation of parameters works, but it is unnecessarily elaborate in this situation. Let

$$
\begin{aligned}
y &= u(x) \ln x \\
y' &= u'(x) \ln x + u(x) \frac{1}{x} \\
y'' &= u''(x) \ln x + 2u'(x) \frac{1}{x} - u(x) \frac{1}{x^2}.
\end{aligned}
$$

Substituting into the differential equation we obtain

$$x^2u''(x)\ln x + 2xu'(x) - u(x) + xu'(x)\ln x + u(x) = x^2 \ln x u''(x) + x(2+\ln x)u'(x) = 0.$$

Letting $v = u'$ we obtain the separable equation

$$\frac{dv}{v} = -\frac{2+\ln x}{x \ln x}\, dx.$$

Integrating and making the substitution $w = \ln x$, $dw = dx/x$,

$$\ln v = -\int \frac{2+\ln x}{x\ln x}\, dx = -\int \frac{2+w}{w}\, dw = -2\ln w - w = -2\ln(\ln x) - \ln x.$$

(You don't need to keep track of constants of integration when doing variation of parameters. Also, you could have left the integral unevaluated if you were short on time.) Exponentiating,

$$v = (\ln x)^{-2}\frac{1}{x}.$$

Integrating, making the substitution $z = \ln x$, $dz = (1/x)\, dx$

$$u = \int v\, dx = \int z^{-2}\, dz = -\frac{1}{z} = -\frac{1}{\ln x}.$$

Finally, a second solution to the given equation is

$$y_2 = u(x)\ln x = -1$$

and a fundamental system of solutions is $\ln x$, -1.

4. The homogenenous equation has general solution $y_h = c_1\cos\omega x + c_2 \sin\omega x$, which is also the solution to the equation $y'' + \omega^2 y = g(x)$ on the interval $0 < x$. For a particular solution on the interval $x < 0$, match the initial condition to obtain $c_1 = c_2 = 0$ so $y = 0$ is the solution on that interval. For a particular solution on the interval $0 \le x \le 4\pi/\omega$ we use undetermined coefficients. We look for a solution of the form

$$\begin{aligned} y_p &= Ax\cos\omega x + Bx\sin\omega x + C\cos\omega x + D\sin\omega x \\ y_p' &= -A\omega x\sin\omega x + B\omega x\cos\omega x + (A + D\omega)\cos\omega x + (B - C\omega)\sin\omega x \\ y_p'' &= -A\omega^2 x\cos\omega x - B\omega^2 x\sin\omega x + (2B\omega - C\omega^2)\cos\omega x + (-2A\omega - D\omega^2)\sin\omega x. \end{aligned}$$

Substituting into the differential equation gives

$$2B\omega\cos\omega x - 2A\omega\sin\omega x = F_0\sin\omega x$$

So $B = 0$, $A = -F_0/2$, and there is no condition on C and D so we can set them equal to 0. (We could have saved ourselves a bit of work by noticing that $\cos\omega x$ and $\sin\omega x$ are in the kernel of the operator so they cannot contribute to the particular solution.) So we have general solution on the interval $0 \le x \le 4\pi/\omega$ of

$$y = c_1\cos\omega x + c_2\sin\omega x - \frac{F_0}{2}x\cos\omega x.$$

Matching with the initial condition,

$$\begin{aligned} y(0) &= c_1 = 0 \\ y'(0) &= c_2 - \frac{F_0}{2} = 0 \end{aligned}$$

implies the particular solution on the interval $0 \le x \le 4\pi/\omega$ is $y = (F_0/2)\sin\omega x - (F_0/2)x\cos\omega x$. Note that at the endpoint of the interval $y(4\pi/\omega) = -2\pi F_0/\omega$ and $y'(4\pi/\omega) = c_2 - (F_0/2) = F_0/2 - F_0/2 = 0$.

In summary, the problem becomes an initial value problem on the interval $x > 4\pi/\omega$ with general solution $y = c_1 \cos \omega x + c_2 \sin \omega x$ and initial conditions $y(2\pi/\omega) = -2\pi F_0/\omega$, $y'(4\pi/\omega) = 0$ which imply $c_1 = -2\pi F_0/\omega$, $c_2 = 0$.

Altogether, the solution to the equation is

$$y = \begin{cases} 0, & x < 0 \\ (F_0/2)\sin \omega x - (F_0/2)x \cos \omega x, & 0 \le x \le 4\pi/\omega \\ -(2\pi F_0/\omega)\cos \omega x & x > 4\pi/\omega. \end{cases}$$

You should check that the solution satisfies the differential equation, the initial conditions, and is continuous.

5. Let our differential equation be $Ly = g(t)$. To get started on this we need solutions to the homogeneous equation $Ly = 0$. Note that if y_1 and y_2 are two solutions to the non-homogeneous equation, $Ly_1 = g(t) = Ly_2$ so $L(y_2 - y_1) = 0$. Therefore two solutions to the non-homogeneous equation are $y_2 - y_1 = e^{2t}$ and $y_3 - y_1 = 1 + 2e^{2t}$. Therefore a fundamental system of solutions to the homogeneous equation is 1 and e^{2t}. The operator L must be an annihilator of those two functions, therefore $L = D(D-2)$. Since $y_1 = t^2$ is a solution of the homogeneous equation, we must have $g(t) = L(t^2) = 2 - 4t$, and the differential equation is $y'' - 2y' = 2 - 4t$ with general solution $y = c_1 + c_2 e^{2t} + t^2$.