MATH281 200610 Problem Set 1 Solutions DRAFT

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1. (a) The order is 3 since we take a third derivative. The equation is nonlinear because the nonlinear term $(dy/dx)^4$ appears.
   (b) The order is 2 since we take a second derivative. The equation is nonlinear because the nonlinear term $-k/R^2$ appears.
   (c) The order is 3 since we take a third derivative. The equation is linear since each term is linear.

2. (a) Taking a derivative,
$$\frac{dy}{dt} = 24e^{-20t}.$$
Then the left hand side of the differential equation becomes
$$\frac{dy}{dt} + 20y = 24e^{-20t} + 20\left(\frac{6}{5} - \frac{6}{5}e^{-20t}\right) = 24e^{-20t} + 24 - 24e^{-20t} = 24$$
which agrees with the right hand side of the equation, so the given function provides a solution to the equation. The solution is defined and C^1 on the entire real line so we can say $I = (-\infty, \infty)$.
   (b) Differentiating,
$$y' = 25\sec^2 5x.$$
The above gives the left hand side of the equation; the right hand side is $25 + (5\tan 5x)^2 = 25 + 25\tan^2 5x = 25(1 + \tan^2 5x) = 25\sec^2 5x$ by the Pythagorean identity. Since the left hand side and right hand side agree, the given function provides a solution to the equation. The function is defined and continuous on the interval $-\pi/2 < 5x < \pi/2$, for example, so we can take $I = (-\pi/10, \pi/10)$.
   (c) Differentiating,
$$y' = -\frac{1}{2}(1 - \sin x)^{-3/2}\frac{d}{dx}(1 - \sin x) = \frac{1}{2}(1 - \sin x)^{-3/2}\cos x.$$
Substituting into the left hand side L of the differential equation,
$$L = 2y' = (1 - \sin x)^{-3/2}\cos x.$$
Substituting into the right hand side R of the differential equation,
$$R = y^3\cos x = (1 - \sin x)^{-3/2}\cos x.$$
Since $L = R$ the given function satisfies the differential equation.
Now we need an interval on which the function is a solution to the differential equation. The given function is defined and continuous when $1 - \sin x \neq 0$ which is true, for example, on the interval I defined by $-3\pi/2 < x < \pi/2$, so that is an example of an interval I on which the function represents a solution to the differential equation. (Give other examples of such intervals.)

3. (a) Differentiating implicitly,

$$\begin{aligned} d(-2x^2y+y^2) &= d(1) \\ d(-2x^2y)+d(y^2) &= 0 \\ -2d(x^2y)+d(y^2) &= 0 \\ -2\left(d(x^2)y+x^2\,dy\right)+2y\,dy &= 0 \\ -2\left(2x\,dx\,y+x^2\,dy\right)+2y\,dy &= 0 \\ -2xy\,dx+(x^2-y)dy &= 0 \end{aligned}$$

so the given implicit function defines a solution to the differential equation. Solving the quadratic equation $y^2-2x^2y-1=0$ for y gives $y=x^2\pm\sqrt{x^4+1}$; for a solution we consistently choose either plus or minus, in which case the function is defined and continuous for all x in the interval $I=(-\infty,\infty)$.

(b) Differentiating,

$$\frac{dy}{dx}=-2xe^{-x^2}\int_0^x e^{t^2}\,dt+e^{-x^2}\frac{d}{dx}\int_0^x e^{t^2}\,dt-2c_1xe^{-x^2}=-2xe^{-x^2}\int_0^x e^{t^2}\,dt+e^{-x^2}e^{x^2}-2c_1xe^{-x^2}$$

by the Fundamental Theorem of Calculus. Substituting the known values for y and y' into the left hand side of the equation gives

$$-2xe^{-x^2}\int_0^x e^{t^2}+1-2c_1xe^{-x^2}+2xe^{-x^2}\int_0^x e^{t^2}+2c_1xe^{-x^2}=1$$

which agrees with the right hand side, so the given function is a solution to the given equation. The given function is defined and continuous for all x in $I=(-\infty,\infty)$ by the Fundamental Theorem of Calculus.

(c) Since the equation is third order we need to take 3 derivatives:

$$\begin{aligned} y' &= -c_1x^{-2}+c_2+c_3\ln x+c_3+8x \\ y'' &= 2c_1x^{-3}+c_3x^{-1}+8 \\ y''' &= -6c_1x^{-4}-c_3x^{-2}. \end{aligned}$$

Substituting what we know about y into the left hand side L of the given equation gives

$$\begin{aligned} L &= x^3(-6c_1x^{-4}-c_3x^{-2})+2x^2(2c_1x^{-3}+c_3x^{-1}+8)-x(-c_1x^{-2}+c_2+c_3\ln x+c_3+8x) \\ &\quad +c_1x^{-1}+c_2x+c_3x\ln x+4x^2 \\ &= -6c_1x^{-1}-c_3x+4c_1x^{-1}+2c_3x+16x^2+c_1x^{-1}-c_2x-c_3x\ln x-c_3x-8x^2 \\ &\quad +c_1x^{-1}+c_2x+c_3x\ln x+4x^2 \\ &= 12x^2 \end{aligned}$$

after all the dust has cleared, which agrees with the right hand side, so the given function satisfies the given equation. The function is defined and continuous on the interval $I=(0,\infty)$ which is the largest interval on which a solution may be defined.

4. Because the given function agrees with a polynomial function at any non-zero point on the line, it is differentiable with continuous derivative (i.e., continuously differentiable at such points). To check differentiability at 0 we need to use the definition of derivative: y is differentiable at 0 if and only if the limit

$$l=\lim_{h\to 0}\frac{y(0+h)-y(0)}{h}$$

exists. To evaluate the limit, we will find it useful to consider left and right-sided limits. From the right,

$$l^+ = \lim_{h\to 0^+} \frac{y(0+h)-y(0)}{h} = \lim_{h\to 0^+} \frac{h^2-0}{h} = \lim_{h\to 0^+} h = 0,$$

and from the left,

$$l^- = \lim_{h\to 0^-} \frac{y(0+h)-y(0)}{h} = \lim_{h\to 0^+} \frac{-h^2-0}{h} = \lim_{h\to 0^+} -h = 0.$$

Since the left and right-sided limits agree, the two-sided limit exists and has the value equal to that of the one-sided limits, i.e., we can conclude that y is differentiable at 0 with derivative $l^+ = l^- = 0$. Therefore the derivative of y is given by

$$y'(x) = \begin{cases} -2x, & x < 0 \\ 0, & x = 0 \\ 2x, & x > 0. \end{cases}$$

It is readily seen that $y'(x)$ is a continuous function (in fact, $y'(x) = 2|x|$) so y is continuously differentiable on the entire real line. Substituting what we know about y into the left-hand side L of the differential equation we obtain

$$xy' - 2y = \begin{cases} x(-2x) - 2(-x^2) = 0, & x < 0 \\ 0(0) - 2(0) = 0, & x = 0 \\ x(2x) - 2(x^2) = 0, & x > 0 \end{cases}$$

i.e., the differential equation is satisfied in all three cases, so we have found a solution defined on the interval $-\infty < x < \infty$.

In fact, every member of the two-parameter family of functions

$$y = \begin{cases} c_1 x^2, & x < 0 \\ c_2 x^2, & x \geq 0 \end{cases}$$

(where c_1 and c_2 are constants) is a solution to the differential equation, as you should be able to check for yourself.

5. Taking derivatives,

$$\begin{aligned} y' &= mx^{m-1} \\ y'' &= m(m-1)x^{m-2}. \end{aligned}$$

(a) $xy'' + 2y' = 0$ Substituting the above into the equation we obtain

$$\begin{aligned} xm(m-1)x^{m-2} + 2mx^{m-1} &= 0 \\ m(m-1)x^{m-1} + 2mx^{m-1} &= 0 \end{aligned}$$

Assuming that $x \neq 0$ (which means that our interval I may be restricted to $(0,\infty)$ or $(-\infty, 0)$) we can divide through by x^{m-1} to obtain the quadratic equation

$$m^2 + m = 0$$

which has solutions $m = 0$, $m = -1$. Both values of m give solutions defined on the appropriate intervals ($I = (-\infty,\infty)$ for $m = 0$ and $I = (0,\infty)$, for example, for $m = -1$).

(b) Again, substituting the above derivatives into the given equation we find that to have a solution we must have

$$L = x^2 m(m-1)x^{m-2} - 7xmx^{m-1} + 15x^m = m(m-1)x^m - 7mx^m + 15x^m = 0.$$

Dividing the latter equation through by x^m we obtain the quadratic equation

$$m^2 - 8m + 15 = 0$$

with solutions $m = 3$ and $m = 5$. Both values of m give solutions to the differential equation defined on the entire real line.