MATH281 200610 Problem Set 3 Solutions DRAFT

Edward Doolittle

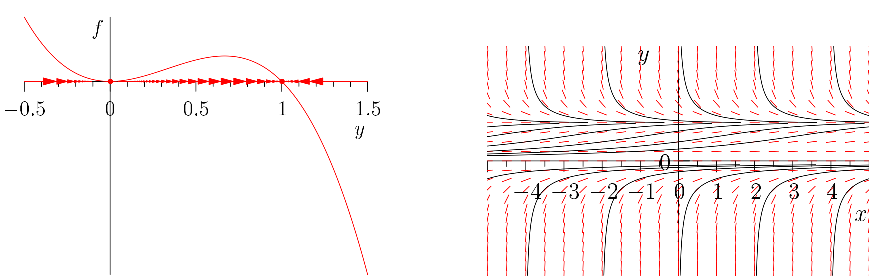
Wednesday, February 1, 2006

1. The critical points are where the differential equation is of the form $dy/dx = 0$, i.e., the points where the left hand sides of the given equations are zero.

   The phase portraits are constructed by putting arrows on the y-axis with the property that the arrow rooted at y should have length proportional to $f(y)$; negative values of $f(y)$ give arrows pointing to the left, and positive values of $f(y)$ give arrows pointing to the right. To avoid clutter we don't put too many arrows on the axis; a good strategy is to start an arrow where the previous arrow ended. We also mark the important critical points in the phase portrait with dots (arrows that have a length of 0). We also should avoid arrows crossing over critical points, which makes it look like a trajectory can pass through a critical point (which is never possible; why not?), so we shrink our arrows (hopefully all by the same amount) so they never cross equilibrium points.

   Solution curves can be constructed on the xy-plane by first splitting the plane by the equilibrium solutions which are represented by horizontal lines in the xy-plane. The solution curves are confined into the regions bounded by the equilibrium solution lines (why?). We then graph the direction field on the xy-plane, and approximate the solution curves which are compatible with the direction field. We only have to guess one solution curve in each region for an autonomous DE because all other curves in a region bounded by equilibrium solution can be obtained by translation.

   (a) $y^2 - y^3 = 0 \implies y = 0, 1$. The critical point $y = 0$ is unstable because $y^2 - y^3 > 0$ for small positive values of y which shows that a solution which starts close to 0 on the positive side increases away from 0. (The point $y = 0$ is actually a 'semi-stable' equilibrium point because it's stable on one side, but semi-stable is still unstable.) The critical point $y = 1$ is stable because $y^2 - y^3 > 0$ for values slightly less than $y = 1$, showing that values slightly less than $y = 1$ get pushed closer; on the other hand, values of y slightly greater than $y = 1$ get pushed closer to 1 also because $y^2 - y^3 < 0$ for such y.) See the phase portrait in Figure 1. (Those blobs are



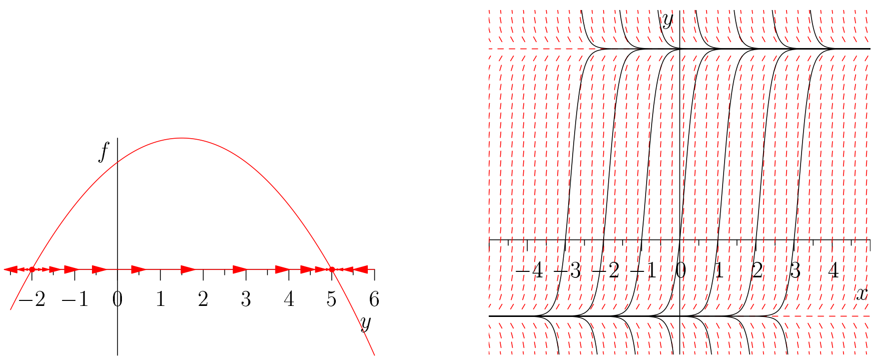
(a) Phase portrait (b) Direction field in time domain

Figure 1: Qualitative analysis of $\dfrac{dy}{dx} = y^2 - y^3$

   supposed to be arrowheads pointing to the right if f is positive and to the left if f is negative; in most cases the arrow is so short that the body of the arrow is missing. You might try zooming in on the diagrams with your PDF reader. Suggestions for improving the artistry of my diagrams are most welcome.)

The direction field and sample solution curves are also given in Figure 1. Typically one would draw the direction field using a computer and then approximate solution curves by hand. I drew the solution curves by actually solving the equation; in general that is very difficult, however, and not necessary when doing a qualitative analysis.

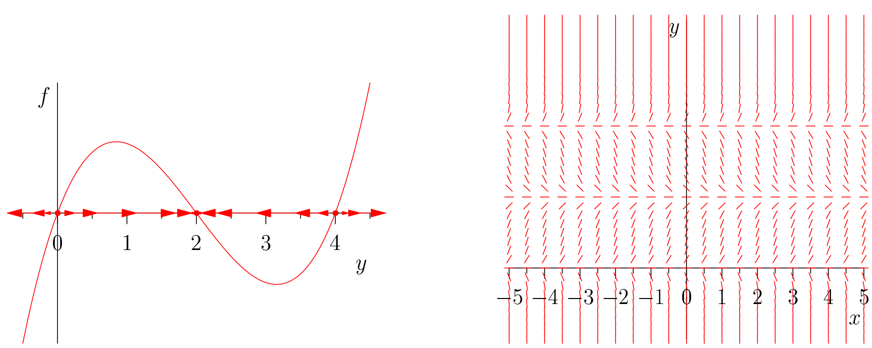
(b) Factoring, $10 + 3y - y^2 = (5 - y)(2 + y)$, so the critical points are at $y = 5$ and $y = -2$. The function $10 + 3y - y^2$ increases through $y = -2$ (why?) which shows that $y = -2$ is an unstable critical point of the differential equation (why?). Similarly, at $y = 5$, the function $10 + 3y - y^2$ decreases through the critical point which shows that it is a stable critical point. See Figure 2. I have filled in a few solution curves which I obtained by solving the equation, but you normally wouldn't solve the equation in a qualitative analysis.



(a) Phase portrait (b) Direction field in time domain

Figure 2: Qualitative analysis of $\dfrac{dy}{dx} = 10 + 3y - y^2$

(c) The critical points are $y = 0$, $y = 2$, and $y = 4$. The points $y = 0$ and $y = 4$ are unstable and $y = 2$ is stable by the criterion established in the previous problem. See Figure 3. I have not drawn any solution curves; you should try drawing some curves onto the direction field yourself.



(a) Phase portrait (b) Direction field in time domain

Figure 3: Qualitative analysis of $\dfrac{dy}{dx} = y(2 - y)(4 - y)$

2. We follow the same process for graphing the phase portrait as in question 1. Note that we do not need to know the exact formula for the function f to construct a qualitatively-correct phase portrait.

   (a) This problem is just a variation on 1(b). The exact nature of the function $f(y)$ doesn't matter for the qualitative analysis, only the 'shape', i.e., the points where it crosses or touches the x-axis and the sign of the function between those points. The equilibrium solutions are $y = 0$ and $y = c$. The former is unstable because arrows near 0 are pointing away from 0; the latter is stable because arrows near c are pointing toward c.

(b) This problem is just a variation on 1(c). The same comments apply as for 2(a). The equilibrium solutions are approximately $y = -2.1$, $y = 0.5$, and $y = 1.6$. The equilibrium solution at $y = 0.5$ is stable because arrows near 0.5 are pointed towards 0.5; the other two equilibrium solutions are unstable.

3. (a) Separating variables,

$$\frac{1}{(y-1)^2}dy = dx$$
$$\int \frac{1}{(y-1)^2}dy = \int dx = x + c.$$

To integrate the y integral, make the change of variables $u = y - 1$ which leads to

$$\int \frac{1}{(y-1)^2}dy = \int u^{-2}\, du = -u^{-1} = -\frac{1}{y-1}.$$

The general solution to the equation is therefore

$$-\frac{1}{y-1} = x + c$$
$$y - 1 = -\frac{1}{x+c}$$
$$y = 1 - \frac{1}{x+c}.$$

You should check the solution by substituting into the differential equation and identifying an interval I on which the solution is C^1. Is the equilibrium solution $y = 1$ lost?

(b) You need to do some algebra to see that the equation is separable. The right-hand side is

$$e^{-y} + e^{-2x-y} = e^{-y} + e^{-2x}e^{-y} = e^{-y}(1 + e^{-2x})$$

so the equation can be written

$$e^x y \frac{dy}{dx} = e^{-y}(1 + e^{-2x})$$
$$e^y y\, dy = e^{-x}(1 + e^{-2x})dx$$
$$\int e^y y\, dy = \int (e^{-x} + e^{-3x})dx$$

The x integral is straightforward:

$$\int (e^{-x} + e^{-3x})dx = -e^{-x} - \frac{1}{3}e^{-3x} + c.$$

The y integral should be done by integration by parts with $u = y$, $dv = e^y\, dy$, $du = dy$, $v = e^y$:

$$\int e^y y\, dy = ye^y - \int e^y\, dy = ye^y - e^y.$$

The solution in implicit form is

$$(y-1)e^y = -e^{-x} - \frac{1}{3}e^{-3x} + c.$$

The above implicit relation is difficult or impossible to solve for y explicitly, so you should just leave it in that form. The implicit function theorem of second-year calculus guarantees the existence of the interval I under most circumstances. You should use implicit differentiation to check your solution, however. Do you think there may be any lost solutions?

(c) Separating variables,

$$\begin{aligned}
\frac{y}{(1+y^2)^{1/2}}dy &= \frac{x}{(1+x^2)^{1/2}}dx \\
(1+y^2)^{1/2} &= (1+x^2)^{1/2} + c \\
1+y^2 &= \left((1+x^2)^{1/2}+c\right)^2 \\
y^2 &= \left((1+x^2)^{1/2}+c\right)^2 - 1 \\
y &= \pm\left(\left((1+x^2)^{1/2}+c\right)^2 - 1\right)^2
\end{aligned}$$

You should check the answer. Do you think there may be any lost solutions?

4. We first find general solutions of the differential equations and then fit them to the initial value problem by an appropriate choice of constants.

(a) Separating variables and integrating by partial fractions as in the lectures,

$$\begin{aligned}
\frac{dy}{y^2-1} &= \frac{dx}{x^2-1} \\
\frac{1}{2}\int\left(\frac{1}{y-1}-\frac{1}{y+1}\right)dy &= \frac{1}{2}\int\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx \\
\ln\left|\frac{y-1}{y+1}\right| &= \ln\left|\frac{x-1}{x+1}\right| + c.
\end{aligned}$$

Exponentiating both sides and dropping the absolute value signs,

$$\frac{y-1}{y+1} = \pm k\frac{x-1}{x+1},$$

where $k > 0$. The positive parameter k can be replaced by a parameter k with no restrictions (careful about the case $k = 0$ which returns a lost solution).

We could now continue to solve for y, but we can save ourselves a bit of headache by substituting the initial condition into the implicit general solution. Setting $x = 2$ and $y = 2$,

$$\begin{aligned}
\frac{1}{3} &= k\frac{1}{3} \\
k &= 1.
\end{aligned}$$

Now solving for y gives the particular solution $y = x$. You should check that that is indeed a solution to the initial value problem; in particular, you should identify an interval on which the solution makes sense; the interval shouldn't include $x = 1$ (why not?).

(b) The given equation is a non-homogeneous linear equation, but we can actually solve this one using only the techniques from chapter 2.2. Separating variables,

$$\begin{aligned}
\frac{dy}{dt} &= 1-2y \\
\frac{dy}{1-2y} &= dt \\
-\frac{1}{2}\ln|1-2y| &= t + c.
\end{aligned}$$

As usual, we can try to solve for the value of c using the initial conditions in the implicit general solution:

$$\begin{aligned}-\frac{1}{2}\ln|1-2(5/2)| &= 0+c\\ -\frac{1}{2}\ln|1-5| &= c\\ -\frac{1}{2}\ln 4 &= c\\ c &= -\ln 2.\end{aligned}$$

You should now find the solution in explicit form, check the solution, and identify an interval on which the solution makes sense.

(c) Separating variables,

$$\begin{aligned}\frac{dy}{1+4y^2} &= -\frac{x}{1+x^4}dx\\ \int\frac{dy}{1+4y^2} &= -\int\frac{x}{1+x^4}dx.\end{aligned}$$

The y integral is a variation on $\tan^{-1}$, while the x integral can be done with the substitution $u = x^2$:

$$\begin{aligned}\frac{1}{2}\tan^{-1}(2y) &= -\frac{1}{2}\int\frac{du}{1+u^2}\\ \tan^{-1}(2y) &= -\tan^{-1}(x^2)+c.\end{aligned}$$

As usual, we make use of the initial condition while the solution is still in implicit form on the chance that there may be some simplification:

$$\begin{aligned}\tan^{-1}(2\cdot 0) &= -\tan^{-1}(1^2)+c\\ 0 &= -\frac{\pi}{4}+c\\ c &= \frac{\pi}{4}.\end{aligned}$$

You should now check your solution. (You know the drill.) A nice explicit formula for y can be found using the addition formula for tan.

5. Use the formula

$$df(x,y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

to obtain

$$\frac{2x}{x^2+10}dx - \csc y\cot y dy = 0.$$

Clearing fractions gives the desired differential equation. Lost solutions will appear when we might be dividing by 0, i.e., when $x^2+10=0$ (never) or when $\sin y = 0$ (at $y = k\pi$). None of the latter can be found from the implicit solution. (Why not? Is it possible to get any constant solution from the implicit solution?)