MATH281 200610 Problem Set 4 Solutions DRAFT

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1. (a) First, divide through by the coefficient of dy/dx to obtain the differential equation

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{3}{x}. \tag{1}$$

Note that the coefficients now are not defined at $x = 0$, so we must solve the equation separately on the intervals $x > 0$ and $x < 0$. Let's assume that $x > 0$; the calculations for $x < 0$ are parallel. There are two ways we can proceed: variation of parameters or us of an integrating factor.

Variation of parameters. The corresponding homogeneous equation is

$$\frac{dy}{dx} + \frac{2}{x}y = 0$$

which is separable:

$$\begin{aligned} \frac{dy}{y} &= -\frac{2\,dx}{x} \\ \int \frac{dy}{y} &= \int \frac{2\,dx}{x} \\ \ln|y| &= -2\ln|x| + c \\ y &= kx^{-2}. \end{aligned}$$

To find a solution to the non-homogeneous equation, we vary the parameter k, i.e., we search for a solution to the original equation of the form $y = k(x)x^{-2}$. Substituting $y = k(x)x^{-2}$ into the original differential equation we obtain

$$\begin{aligned} x(k'(x)x^{-2} - 2x^{-3}k(x)) + 2k(x)x^{-2} &= 3 \\ k'(x)x^{-1} &= 3 \\ k'(x) &= 3x \\ k(x) &= \frac{3}{2}x^2, \end{aligned}$$

where we can ignore the constant of integration in this step. That value of $k(x)$ gives a particular solution $y_p(x) = k(x)x^{-2} = 3/2x^2x^{-2} = 3/2$. It follows that the general solution is the sum of the particular solution and the general solution $y_h = kx^{-2}$ of the homogeneous equation, i.e., the general solution to the differential equation is

$$y(x) = \frac{3}{2} + kx^{-2}.$$

You should find a family of solutions on $x < 0$.

Integrating factor. The function $k(x)$ identified above may be used as an integrating factor. In fact, we can multiply $k(x)$ by any non-zero constant to obtain another integrating factor; the simplest choice is the function $q(x) = x^2$. We multiply (1) (not the original equation) by x^2 to obtain

$$x^2 \frac{dy}{dx} + 2xy = 3x$$

which we should recognize as

$$\frac{d}{dx}(x^2 y) = 3x.$$

Integrating both sides we obtain

$$\begin{aligned} x^2 y &= \frac{3}{2}x^2 + c \\ y &= \frac{3}{2} + cx^{-2}, \end{aligned}$$

identical to the family of solutions found by variation of parameters. Again, there is an implicit assumption that we are in the interval $x > 0$ or $x < 0$.

(b) It may be difficult to recognize this equation as linear. There is a term of the form ye^y, so it can't be linear in y, so let's try solving for x in terms of y. Divinding through by dy we get

$$y\frac{dx}{dy} + 2x = ye^y.$$

The differential operator on the left hand side is identical in form to the operator of the previous problem, so the same integrating factor should work. Multiply the equation through by y to obtain

$$\begin{aligned} y^2\frac{dx}{dy} + 2yx &= y^2 e^y \\ \frac{d}{dy}(y^2 x) &= y^2 e^y \\ y^2 x &= \int y^2 e^y \, dy + c \\ x &= y^{-2} \int y^2 e^y \, dy + cy^{-2}. \end{aligned}$$

The integral on the right hand side may be evaluated by integration by parts with $u = y^2$, $dv = e^y$, $du = 2y\, dy$, $v = e^y$:

$$\int y^2 e^y \, dy = y^2 e^y - \int 2ye^y \, dy.$$

Again, with $u = y$, $dv = e^y$, $du = dy$, $v = e^y$:

$$\int 2ye^y \, dy = 2ye^y - 2\int e^y dy = 2ye^y - 2e^y$$

so

$$\int y^2 e^y \, dy = y^2 e^y - 2ye^y + 2e^y$$

and

$$x = e^y - 2y^{-1}e^y + 2y^{-2}e^y + cy^{-2}$$

is the general (implicit) solution to the original equation. (Check.)

(c) We can streamline our calculations by recognizing that the function $e^{\int P(x)\, dx}$ is always an integrating factor for the linear differential operator $d/dx + P(x)$. Putting our equation into standard form by dividing through by $\tan x$, we have

$$\begin{aligned} P(x) &= \cot x \\ \int P(x) \, dx &= \int \cot x \, dx = \ln|\sin x| \\ e^{\int P(x)\, dx} &= |\sin x|, \end{aligned}$$

where we ignore the constant of integration in the evaluation of the integrating factor. Restricting attention to the interval $0 < x < \pi/2$, we have an integrating factor $\sin x$; multiplying the equation in standard form through by $\sin x$ we obtain the equation

$$\begin{aligned}(\sin x)y' + (\cos x)y &= \sec^2 x\\ \frac{d}{dx}((\sin x)y) &= \sec^2 x\\ (\sin x)y &= \tan x + c\\ y &= \sec x + c\csc x.\end{aligned}$$

You should check that the above functions are solutions on the interval $0 < x < \pi/2$. What about on other intervals?

2. (a) An integrating factor is given by

$$e^{\int(-1/y)\,dy} = e^{-\ln|y|} = \frac{1}{|y|}$$

leading to the general solution on $y > 0$

$$\begin{aligned}\frac{d}{dy}\left(y^{-1}x\right) &= 2\\ y^{-1}x &= 2y + c\\ x &= 2y^2 + cy.\end{aligned}$$

Using the initial condition,

$$\begin{aligned}1 &= 2(2)^2 + c(2)\\ c &= -\frac{7}{2}\end{aligned}$$

giving the solution

$$x = 2y^2 - \frac{7}{2}y.$$

You should check the solution and identify an interval on which the solution makes sense.

(b) We can write the equation

$$\frac{dT}{dt} - kT = -kT_m.$$

An integrating factor is e^{-kt}:

$$\begin{aligned}\frac{d}{dt}(e^{-kt}T) &= -ke^{-kt}T_m\\ e^{-kt}T &= e^{-kt}T_m + c\\ T &= T_m + ce^{kt}.\end{aligned}$$

Substituting $t = 0$ we find $c = T_0 - T_m$.

(c) Multiplying the equation through by $\cos^2 x$ gives $P(x) = \tan x$ and we find the integrating factor

$$e^{\int \tan x\,dx} = e^{\ln|\sec x|} = |\sec x|.$$

Restricting to the interval $-\pi/2 < x < \pi/2$ which contains the initial point $x = 0$, we have the integrating factor $\sec x$ which gives the equation

$$\begin{aligned}(\sec x)y' + (\sec x\tan x)y &= \cos x\\ \frac{d}{dx}((\sec x)y) &= \cos x\\ (\sec x)y &= \sin x + c\\ y &= \sin x\cos x + c\cos x.\end{aligned}$$

The initial condition tells us that $c = 1$. You should check that $y = \sin x \cos x + \cos x$ gives a solution to the initial value problem on the interval $-\pi/2 < x < \pi/2$. Can we extend the interval to a larger interval?

3. (a) An integrating factor for the equation is e^x. On the interval $0 \le x \le 1$ the equation becomes

$$\begin{aligned} \frac{dy}{dx} + y &= 1 \\ e^x \frac{dy}{dx} + e^x y &= e^x \\ \frac{d}{dx}(e^x y) &= e^x \end{aligned}$$

with general solution

$$\begin{aligned} e^x y &= e^x + c_1 \\ y &= 1 + c_1 e^{-x}. \end{aligned}$$

Similarly, on the interval $1 < x$ the equation becomes

$$\begin{aligned} \frac{dy}{dx} + y &= -1 \\ e^x \frac{dy}{dx} + e^x y &= -e^x \\ \frac{d}{dx}(e^x y) &= -e^x \end{aligned}$$

with general solution

$$\begin{aligned} e^x y &= -e^x + c_2 \\ y &= -1 + c_2 e^{-x}. \end{aligned}$$

To satisfy the initial condition $y(0) = 1$ (with $x_0 = 0$ in the interval $0 \le x \le 1$) we must have

$$1 = y(0) = 1 + c_1 e^{-0} = 1 + c_1 \implies c_1 = 0,$$

so on $0 \le x \le 1$ the particular solution is $y(x) = 1$. That implies that $y(1) = 1$. In order for the solution to be continuous, we must then have

$$1 = y(1) = \lim_{x \to 1^+} y(x) = -1 + c_2 e^{-1} = -1 + \frac{c_2}{e} \implies c_2 = 2e.$$

Altogether, our particular solution is

$$y(x) = \begin{cases} 1, & 0 \le x \le 1 \\ -1 + 2e^{1-x}, & 1 < x. \end{cases}$$

(b) An integrating factor for the operator $d/dx + 2x/(1+x^2)$ is $e^{\int 2x/(1+x^2)\,dx} = e^{\ln(1+x^2)} = 1 + x^2$, so, oops, the equation is already in a form that can be integrated. On the interval $0 \le x < 1$ the equation becomes

$$\begin{aligned} (1+x^2)\frac{dy}{dx} + 2xy &= x \\ \frac{d}{dx}((1+x^2)y) &= x \end{aligned}$$

with general solution

$$(1+x^2)y = \frac{1}{2}x^2 + \frac{1}{2}c_1$$
$$y = \frac{c_1 + x^2}{2(1+x^2)}.$$

(I just threw the factor of 1/2 in front of c_1 to simplify the final answer; we can play around with the form of the constant.) Similarly, on the interval $1 \le x$ the equation becomes

$$(1+x^2)\frac{dy}{dx} + 2xy = -x$$
$$\frac{d}{dx}((1+x^2)y) = -x$$

with general solution

$$(1+x^2)y = -\frac{1}{2}x^2 - \frac{1}{2}c_2$$
$$y = -\frac{c_2 + x^2}{2(1+x^2)}.$$

To satisfy the initial condition $y(0) = 0$ on the interval $0 \le x < 1$ we must have

$$0 = y(0) = \frac{c_1 + 0^2}{2(1+0^2)} = \frac{c_1}{2} \implies c_1 = 0,$$

so on $0 \le x < 1$ the particular solution is

$$y = \frac{x^2}{2(1+x^2)}.$$

Approaching the right endpoint of the interval we have

$$\lim_{x\to 1^-} y = \frac{(-1)^2}{2(1+(-1)^2)} = \frac{1}{4},$$

so for the solution to be continuous we must have

$$\frac{1}{4} = \lim_{x\to 1^-} y(x) = y(1) = -\frac{c_2 + (1)^2}{2(1+(1)^2)} \implies 1 = -(c_2+1) \implies c_2 = -2.$$

Altogether our particular solution is

$$y(x) = \begin{cases} x^2/(2(1+x^2)), & 0 \le x < 1 \\ (-2+x^2)/(2(1+x^2)), & 1 \le x. \end{cases}$$

4. (a) comparing with the form $df = M(x,y)\,dx + N(x,y)\,dy$, we have

$$M(x,y) = 2x + y$$
$$N(x,y) = x + 6y.$$

Remember that the negative sign goes with N! Taking the appropriate partial derivatives,

$$\frac{\partial M}{\partial y} = 1$$
$$\frac{\partial N}{\partial x} = 1,$$

so the equation is exact. We can assume that there is a function f such that $df = (2x + y)\,dx + (x + 6y)\,dy$. Then f must have the properties

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2x + y \\ f(x, y) &= x^2 + xy + g(y) \\ \frac{\partial f}{\partial y} &= x + g'(y) \\ x + 6y &= x + g'(y) \\ 6y &= g'(y) \\ g(y) &= 3y^2 \ (+C). \end{aligned}$$

Therefore one such f is given by

$$f(x, y) = x^2 + xy + 3y^2$$

and our differential equation can be written

$$d(x^2 + xy + 3y^2) = 0$$

with implicit solution

$$x^2 + xy + 3y^2 = c.$$

An explicit solution could be found by completing the square or otherwise solving a quadratic equation, but we won't bother with that step. You should recognize the solution curves as ellipses, however.

(b) Putting the equation into standard form for exact equations we have

$$\left(1 - \frac{3}{x} + y\right) dx + \left(1 - \frac{3}{y} + x\right) dy = 0.$$

Applying our criterion for exactness,

$$\begin{aligned} \frac{\partial}{\partial y}\left(1 - \frac{3}{x} + y\right) &= 1 \\ \frac{\partial}{\partial x}\left(1 - \frac{3}{y} + x\right) &= 1 \end{aligned}$$

So the equation is exact. Then f must have the properties

$$\begin{aligned} \frac{\partial f}{\partial x} &= 1 - \frac{3}{x} + y \\ f &= x - 3\ln|x| + xy + g(y) \\ \frac{\partial f}{\partial y} &= x + g'(y) \\ 1 - \frac{3}{y} + x &= x + g'(y) \\ g'(y) &= 1 - \frac{3}{y} \\ g(y) &= y - 3\ln|x|, \end{aligned}$$

so we have the implicit solution

$$f(x, y) = x + y - 3\ln|xy| + xy = c.$$

I can't imagine how one might easily solve for y explicitly, but you should at least give some thought as to where the solution is defined.

(c) Here

$$\frac{\partial M}{\partial y} = x + 2y + 1$$
$$\frac{\partial N}{\partial x} = 1;$$

the two are not equal so the equation is not exact. We look for an integrating factor. The integrating factor must satisfy

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$
$$\frac{\partial \mu}{\partial y} M + \mu \frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x} N + \mu \frac{\partial N}{\partial x}.$$

Substituting what we know into the above PDE,

$$y(x + y + 1)\frac{\partial \mu}{\partial y} + (x + 2y + 1)\mu = (x + 2y)\frac{\partial \mu}{\partial x} + \mu.$$

Solving the PDE is hopeless for us unless we make a simplfying assumption. Assuming μ is a function of y alone gives us the ODE

$$y(x + y + 1)\frac{d\mu}{dy} + (x + 2y)\mu = 0$$

which cannot be solved for a function of y alone, so our assumption was not warranted. Now crossing our fingers and assuming that μ is a function of x alone we get

$$(x + 2y)\mu = (x + 2y)\frac{d\mu}{dx}$$

which is solved by $\mu(x) = e^x$, for example. So, multiplying the original equation by that integrating factor and then integrating (the function μN seems easier to integrate so I try that, but in the end the amount of work will be about the same either way), we have

$$\frac{\partial f}{\partial y} = e^x(x + 2y)$$
$$f = e^x xy + e^x y^2 + h(x)$$
$$\frac{\partial f}{\partial x} = e^x xy + e^x y + e^x y^2 + h'(x)$$
$$e^x y(x + y + 1) = e^x(xy + y + y^2) + h'(x)$$
$$h'(x) = 0$$
$$h(x) = 0 \ (+C)$$

giving the implicit solution

$$e^x(xy + y^2) = c.$$

In this case it is possible, but not necessary, to solve for y explicitly.

(d) The equation is not exact, so we ought to look for an integrating factor. We notice that the equation is separable, however, so an integrating factor should be immediate: dividing through by $\sin x$ make the equation exact and also separates the variables. (Every equation with variables separated is exact; why?) You can use either the methods we learned for exact equations or the methods for separable equations (which amount to the same thing, essentially) to obtain the solution

$$-\ln|\sin x| + y + \ln y^2 = c.$$

5. (a) The equation is exact. Solving for f,

$$\begin{aligned}\frac{\partial f}{\partial x} &= e^x + y \\ f &= e^x + xy + g(y) \\ \frac{\partial f}{\partial y} &= x + g'(y) \\ 2 + x + ye^y &= x + g'(y) \\ g'(y) &= 2 + ye^y \\ g(y) &= 2y + (y-1)e^y\end{aligned}$$

where the value of g was obtained by integration by parts. The general solution is

$$e^x + xy + 2y + (y-1)e^y = c.$$

In order to solve the IVP, we substitute the values $x = 0$ and $y = 1$ into the equation to obtain

$$1 + 2 = c \implies c = 3,$$

so the solution to our IVP is given by

$$e^x + xy + 2y + (y-1)e^y = c.$$

(What justifies using the term 'the solution' above?)

(b) Checking the exactness criterion,

$$\begin{aligned}\frac{\partial}{\partial t}(6y^2 - 2t^2) &= -4t \\ \frac{\partial}{\partial y}(ty) &= t,\end{aligned}$$

so the equation is not exact. We should search for an integrating factor. The integrating factor must satisfy the PDE

$$\begin{aligned}\frac{\partial \mu}{\partial t}M + \mu\frac{\partial M}{\partial t} &= \frac{\partial \mu}{\partial y}N + \mu\frac{\partial N}{\partial y}. \\ (6y^2 - 2t^2)\frac{\partial \mu}{\partial t} - 4t\mu &= (ty)\frac{\partial \mu}{\partial y} + t\mu.\end{aligned}$$

Assuming that μ is a function of t alone doesn't work, but assuming that μ is a function of y alone does:

$$\begin{aligned}-4t\mu &= ty\frac{d\mu}{dy} + t\mu \\ y\frac{d\mu}{dy} &= -5\mu\end{aligned}$$

which has one solution $\mu(y) = y^{-5}$. Using that integrating factor and solving for f,

$$\begin{aligned}\frac{\partial f}{\partial t} &= ty^{-4} \\ f(y,t) &= \frac{1}{2}t^2y^{-4} + g(y) \\ \frac{\partial f}{\partial y} &= -2t^2y^{-5} + g'(y) \\ y^{-5}(6y^2 - 2t^2) &= -2t^2y^{-5} + g'(y) \\ 6y^{-3} &= g'(y) \\ g(y) &= -3y^{-2}\end{aligned}$$

giving the general solution

$$\frac{1}{2}t^2y^{-4} - 3y^{-2} = c.$$

Substituting the intial conditions $t = 1$, $y = 1$ we have

$$\frac{1}{2} - 3 = c \implies c = -\frac{5}{2}$$

which gives the solution to the initial value problem

$$t^2 - 6y^2 + 5y^4 = 0.$$

You should check that solution.

(c) Checking the exactness criterion (don't forget to put the equation in standard form $M(x,y)\,dx + N(x,y)\,dy = 0$),

$$\begin{aligned}\frac{\partial M}{\partial y} &= 2y \\ \frac{\partial N}{\partial x} &= -y,\end{aligned}$$

so the equation is not exact. Searching for an integrating factor,

$$\begin{aligned}\frac{\partial \mu}{\partial y}M + \mu\frac{\partial M}{\partial y} &= \frac{\partial \mu}{\partial x}N + \mu\frac{\partial N}{\partial x} \\ (x^2 + y^2 - 5)\frac{\partial \mu}{\partial y} + 2y\mu &= -(y + xy)\frac{\partial \mu}{\partial x} - y\mu.\end{aligned}$$

Assuming μ depends on y alone doesn't help, so we assume μ depends on x alone:

$$\begin{aligned}(y + xy)\frac{d\mu}{dx} + 3y\mu &= 0 \\ (1 + x)\frac{d\mu}{dx} + 3\mu &= 0 \\ \frac{d\mu}{\mu} &= -\frac{3}{1 + x}\,dx \\ \ln|\mu| &= -3\ln|1 + x| \\ \mu &= (1 + x)^{-3}\end{aligned}$$

where we ignore the constant of integration that would normally appear when solving a seperable equation. Multiplying the original equation by the integrating factor gives us the exact equation in standard form

$$(1 + x)^{-3}(x^2 + y^2 - 5)\,dx - (1 + x)^{-3}(y + xy)\,dy = 0$$

so f must satisfy

$$\begin{aligned}\frac{\partial f}{\partial y} &= -(1 + x)^{-2}y \\ f &= -\frac{1}{2}(1 + x)^{-2}y^2 + g(x) \\ \frac{\partial f}{\partial x} &= (1 + x)^{-3}y^2 + g'(x) \\ (1 + x)^{-3}(x^2 + y^2 - 5) &= (1 + x)^{-3}y^2 + g'(x) \\ g'(x) &= (1 + x)^{-3}(x^2 - 5) \\ g(x) &= \int \frac{x^2 - 5}{(1 + x)^3}\,dx.\end{aligned}$$

We might have some difficulty integrating the above function to find g. When we learn more about partial fractions we will learn a systematic procedure for handling such integrals, but for now let's just change variables $u = 1 + x$, $du = dx$:

$$\begin{aligned} g(x) &= \int \frac{(u-1)^2 - 5}{u^3}\, du \\ &= \int \frac{u^2 - 2u - 4}{u^3}\, du \\ &= \int \frac{1}{u} - \frac{2}{u^2} - \frac{4}{u^3}\, du \\ &= \ln|u| + \frac{2}{u} + \frac{2}{u^2} \\ &= \ln|1+x| + \frac{2}{1+x} + \frac{2}{(1+x)^2}. \end{aligned}$$

It would be a good idea to check the integration at this point. It follows that the general solution to the exact equation is

$$-\frac{y^2}{2(1+x)^2} + \ln|1+x| + \frac{2}{1+x} + \frac{2}{(1+x)^2} = c.$$

The initial condition $x = 0$, $y = 1$ then tells us

$$-\frac{1}{2} + 0 + 2 + 2 = c \implies c = \frac{7}{2}$$

which gives the particular solution

$$y^2 = 2(1+x)^2 \ln|1+x| + 4(1+x) + 4 - 7(1+x)^2.$$

You should check the solution. Also, in this case it shouldn't be too hard to identify an interval on which the solution makes sense.

(d) The equation is exact. (Don't forget to put it in standard form.) Solving for f,

$$\begin{aligned} \frac{\partial f}{\partial x} &= y(y + \sin x) \\ f &= xy^2 - y\cos x + g(y) \\ \frac{\partial f}{\partial y} &= 2xy - \cos x + g'(y) \\ -\frac{1}{1+y^2} - \cos x + 2xy &= 2xy - \cos x + g'(y) \\ g'(y) &= -\frac{1}{1+y^2} \\ g(y) &= -\tan^{-1} y \end{aligned}$$

giving the general solution

$$xy^2 - y\cos x - \tan^{-1} y = c.$$

Substituting the initial condition $x = 0$, $y = 1$ into the general solution,

$$-1 - \frac{\pi}{4} = c$$

so the solution to the initial value problem is

$$xy^2 - y\cos x - \tan^{-1} y = -1 - \frac{\pi}{4}.$$

You should check that solution. How do you know there is an interval around $x = 0$ on which we can find (at least theoretically) an explicit solution?