MATH281 200610 Problem Set 5 Solutions DRAFT

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1. (a) Taking derivatives on the interval I,

$$\begin{aligned} y' &= 4c_1e^{4x} - c_2e^{-x} \\ y'' &= 16c_1e^{4x} + c_2e^{-x}. \end{aligned}$$

Substituting into the differential equation, we have on the interval I,

$$\begin{aligned} Ly &= y'' - 3y' - 4y \\ &= 16c_1e^{4x} + c_2e^{-x} - 12c_1e^{4x} + 3c_2e^{-x} - 4c_1e^{4x} - 4c_2e^{-x} \\ &= 0, \end{aligned}$$

so the given functions are solutions to the differential equation on the interval I. To solve the initial value problem, note that

$$\begin{aligned} 1 &= y(0) = c_1e^{4(0)} + c_2e^{-0} = c_1 + c_2 \\ 2 &= y'(0) = 4c_1e^{4(0)} - c_2e^{-0} = 4c_1 - c_2 \end{aligned}$$

Solving the linear system of equations, we have

$$\begin{aligned} c_1 &= \frac{3}{5} \\ c_2 &= \frac{2}{5} \end{aligned}$$

giving the solution

$$y = \frac{3}{5}e^{4x} + \frac{2}{5}e^{-x}$$

to the initial value problem. (Check!)

(b) Taking derivatives,

$$\begin{aligned} y' &= -c_2 \sin x + c_3 \cos x \\ y'' &= -c_2 \cos x - c_3 \sin x \\ y''' &= c_2 \sin x - c_3 \cos x. \end{aligned}$$

Substituting into the differential equation,

$$\begin{aligned} Ly &= y''' + y' \\ &= c_2 \sin x - c_3 \cos x - c_2 \sin x + c_3 \cos x \\ &= 0 \end{aligned}$$

so the given family of functions satisfies the differential equation. To solve the initial value problem, note that the c_is must satisfy the linear system

$$\begin{aligned} 0 &= y(\pi) = c_1 + c_2 \cos \pi + c_3 \sin \pi \\ 2 &= y'(\pi) = -c_2 \sin \pi + c_3 \cos \pi \\ -1 &= y''(\pi) = -c_2 \cos \pi - c_3 \sin \pi, \end{aligned}$$

i.e.,

$$\begin{aligned} 0 &= c_1 - c_2 \\ 2 &= -c_3 \\ -1 &= c_2, \end{aligned}$$

which has the solution $c_1 = -1$, $c_2 = -1$, $c_3 = -2$, giving the solution

$$y = -1 - \cos x - 2\sin x$$

to the initial value problem. (Check!)

(c) Taking derivatives,

$$\begin{aligned} y' &= 2c_2 x \\ y'' &= 2c_2. \end{aligned}$$

Substituting into the differential equation,

$$\begin{aligned} Ly &= xy'' - y' \\ &= x2c_2 - 2c_2 x \\ &= 0, \end{aligned}$$

so the given family of functions satisfies the differential equation. To solve the initial value problem, note that

$$\begin{aligned} 0 &= y(0) = c_1 + c_2(0)^2 \\ 0 &= y'(0) = 2c_2(0) \end{aligned}$$

with the solution $c_1 = 0$, c_2 arbitrary, giving the family of solutions

$$y = c_2 x^2$$

to the initial value problem. (Check!) Why does this not violate the uniqueness provision of Theorem 4.1?

2. (a) By 1(a) the general solution is $y = c_1 e^{4x} + c_2 e^{-x}$. The first boundary condition implies $c_1 + c_2 = 0$, i.e., $c_2 = -c_1$, i.e., the solution must be of the form $y = c_1(e^{4x} - e^{-x}$. The second boundary condition then gives

$$c_1(e^4 - e^{-1}) = \sinh(5/2) = \frac{e^{5/2} - e^{-5/2}}{2} = \frac{e^{3/2}(e^{5/2} - e^{-5/2})}{2e^{3/2}} = \frac{e^4 - e^{-1}}{2e^{3/2}}$$

which implies $c_1 = e^{-3/2}/2$, i.e., the unique solution to the boundry value problem is $y = e^{4x-3/2}/2 - e^{-x-3/2}/2$. (Check!)

(b) By 1(c), a family of solutions to the differential equation is given by $y = c_1 + c_2 x^2$. The first boundary condition implies that $c_1 = 1$, so the solution has the form $y = 1 + c_2 x^2$. The second boundary condition then implies that $y'(1) = 2c_2(1) = 2c_2 = 6$, which then implies that $c_2 = 3$, giving the solution $y = 1 + 3x^2$.

We must be careful about assuming that the solution is unique, despite the above analysis which gives unique values for c_1 and c_2. The difficulty is that we aren't sure that we have the most general family of solutions because Theorem 4.1 doesn't apply at $x = 0$. Can you see any way around that to show that we really do have a unique solution to the boundary value problem?

3. (a) The Wronskian is

$$W = \begin{vmatrix} 0 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{vmatrix} = 0$$

where the determinant was calculated by expanding in the first column. Therefore the given set of functions is linearly dependent on I. (Any set of functions containing 0 is linearly dependent, so we didn't have to calculate the Wronskian in this case.)

(b) The Wronskian is

$$\begin{aligned} W &= \begin{vmatrix} \cos 2x & 1 & \cos^2 x \\ -2\sin 2x & 0 & -2\cos x \sin x \\ -4\cos 2x & 0 & 2\sin^2 x - 2\cos^2 x \end{vmatrix} \\ &= \begin{vmatrix} \cos 2x & 1 & \cos^2 x \\ -2\sin 2x & 0 & -\sin 2x \\ -4\cos 2x & 0 & -2\cos 2x \end{vmatrix} \\ &= -1 \begin{vmatrix} -2\sin 2x & -2\sin 2x \\ -4\cos 2x & -4\cos 2x \end{vmatrix} = 0, \end{aligned}$$

where the trigonometric identity $\sin 2x = 2\sin x \cos x$ was used, and the determinant was expanded in the second column. This shows that the given family of functions is linearly dependent on I. (Try to find a linear relationship among the functions.)

(c) The Wronskian is

$$\begin{aligned} W &= \begin{vmatrix} e^x & e^{-x} & \sinh x \\ e^x & -e^{-x} & \cosh x \\ e^x & e^{-x} & \sinh x \end{vmatrix} = e^x e^{-x} \begin{vmatrix} 1 & 1 & \sinh x \\ 1 & -1 & \cosh x \\ 1 & 1 & \sinh x \end{vmatrix} \\ &= (-\sinh x - \cosh x) - (\sinh x - \sinh x) + (\cosh x + \sinh x) \\ &= 0, \end{aligned}$$

where the determinant was expanded in the first column. This shows that the given functions are linearly dependent. (Try to find a linear relationship among the functions.)

4. (a) Differentiating the first function $y_1 = e^{x/2}$,

$$\begin{aligned} y_1' &= \frac{1}{2} e^{x/2} \\ y_1'' &= \frac{1}{4} e^{x/2}, \end{aligned}$$

and substituting the above into the differential equation,

$$Ly_1 = 4y_1'' - 4y_1' + y_1 = e^{x/2} - 2e^{x/2} + e^{x/2} = 0,$$

so y_1 satisfies the differential equation. Similarly, differentiating the second function $y_2 = xe^{x/2}$,

$$\begin{aligned} y_2' &= e^{x/2} + \frac{x}{2} e^{x/2} \\ y_2'' &= \frac{1}{2} e^{x/2} + \frac{1}{2} e^{x/2} + \frac{x}{4} e^{x/2} = e^{x/2} + \frac{x}{4} e^{x/2}, \end{aligned}$$

and substituting the above into the differential equation,

$$Ly_2 = 4y_2'' - 4y_2 + y_2 = 4e^{x/2} + xe^{x/2} - 4e^{x/2} - 2xe^{x/2} + xe^{x/2} = 0,$$

so y_2 satisfies the differential equation. Now, checking that the given functions are linearly independent,

$$W = \begin{vmatrix} e^{x/2} & xe^{x/2} \\ \frac{1}{2}e^{x/2} & e^{x/2} + \frac{x}{2}e^{x/2} \end{vmatrix} = e^{x/2}(e^{x/2} + \frac{x}{2}e^{x/2}) - xe^{x/2}\frac{1}{2}e^{x/2} = e^x + \frac{x}{2}e^x - \frac{x}{2}e^x = e^x$$

which is never zero, so the given functions are linearly independent. Therefore the two given functions form a fundamental system of solutions to the equation, and the general solution is $y = c_1 e^{x/2} + c_2 x e^{x/2}$.

(b) Taking derivatives of $y_1 = \cos(\ln x)$,

$$\begin{aligned} y_1' &= -\sin(\ln x)\frac{1}{x} \\ y_1'' &= -\cos(\ln x)\frac{1}{x^2} + \sin(\ln x)\frac{1}{x^2}, \end{aligned}$$

and substituting into the differential equation,

$$Ly_1 = x^2y'' + xy' + y = -\cos(\ln x) + \sin(\ln x) - \sin(\ln x) + \cos(\ln x) = 0,$$

so y_1 satisfies the differential equation. Similarly $y_2 = \sin(\ln x)$ satisfies the differential equation (check!). Taking the Wronskian,

$$W = \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ -\sin(\ln x)\frac{1}{x} & \cos(\ln x)\frac{1}{x} \end{vmatrix} = \frac{1}{x}\left(\cos^2(\ln x) + \sin^2(\ln x)\right) = \frac{1}{x}$$

which is never zero on I, so the given functions are linearly independent and therefore form a fundamental system of solutions. The general solution is $y = c_1\cos(\ln x) + c_2\sin(\ln x)$.

(c) We verified that the given functions are solutions to the differential equation in 1(b) above. To determine linear independence, take the Wronskian

$$W = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} = \sin^2 x + \cos^2 x = 1,$$

so the functions are linearly independent and form a fundamental system of solutions to the equation. The general solution is $y = c_1 + c_2\cos x + c_3\sin x$.

5. (a) Guess $y = 7$ as a particular solution; checking, that solution actually works, so the general solution to the given equation is $y = c_1e^{x/2} + c_2xe^{x/2} + 7$.

(b) Guess $y_p = Ax^2$ as a particular solution, where A is unknown. Differentiating, $y_p' = 2Ax$, $y_p'' = 2A$, and substituting into the differential equation, $x^2(2A) + x(2Ax) + Ax^2 = 5Ax^2 = x^2$, so our guess works with $A = 1/5$. Therefore by question 4(b) the general solution to the given equation is $y = c_1\cos(\ln x) + c_2\sin(\ln x) + x^2/5$.

(c) Guess $y_{p_1} = x$ for the first term: $y_{p_1}''' + y_{p_1}' = 0 + 1 = 1$, so that works. Guess $y_{p_2} = Ae^x$ for the second term: $y_{p_2}''' + y_{p_2}' = Ae^x + Ae^x = 2Ae^x = e^x$, so the guess works with $A = 1/2$, and we have the particular solution $y_p = x + e^x/2$. By 4(c), the general solution to the equation is $y = c_1 + c_2\cos x + c_3\sin x + x + e^x/2$.