MATH281 200610 Problem Set 6 Solutions DRAFT

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1. (a) Taking derivatives,

$$y_1' = \frac{1}{3}e^{x/3}$$
$$y_1'' = \frac{1}{9}e^{x/3}.$$

Substituting into the differential equation,

$$Ly_1 = 6y_1'' + y_1' - y_1 = 6\frac{1}{9}e^{x/3} + \frac{1}{3}e^{x/3} - e^{x/3} = 0,$$

so the function y_1 satisfies the differential equation. To find another solution by reduction of order, let $y_2(x) = u(x)y_1(x) = u(x)e^{x/3}$ for some unknown function u. Taking derivatives,

$$y_2' = u'(x)e^{x/3} + \frac{1}{3}u(x)e^{x/3}$$
$$y_2'' = u''(x)e^{x/3} + \frac{2}{3}u'(x)e^{x/3} + \frac{1}{9}u(x)e^{x/3}.$$

Substituting into the differential equation,

$$\begin{aligned}
Ly_2 &= 6y_2'' + y_2' - y_2 \\
&= 6(u'' + \frac{2}{3}u' + \frac{1}{9}u)e^{x/3} + (u' + \frac{1}{3}u)e^{x/3} - ue^{x/3} \\
&= (6u'' + 5u')e^{x/3} + (6\frac{1}{9}ue^{x/3} + \frac{1}{3}ue^{x/3} - ue^{x/3}) \\
&= (6u'' + 5u')e^{x/3} \qquad = 0.
\end{aligned}$$

Letting $v = u'$ and multiplying through by $e^{-x/3}$ we obtain the first order differential equation

$$6v' + 5v = 0$$

with solution $v = e^{-5x/6}$; therefore a solution for the equation in u is $u = -(6/5)e^{-5x/6}$. We can drop the constant factor (why?) to obtain a solution $u = e^{-5x/6}$, and $y_2 = uy_1 = e^{-5x/6}e^{x/3} = e^{-5x/6}e^{2x/6} = e^{-3x/6} = e^{-x/2}$. (You should check that that really is a solution to the differential equation.) We now have a fundamental system of solutions, so the general solution is $y = c_1e^{x/3} + c_2e^{-x/2}$.

(b) Verification that y_1 is a solution to the differential equation is straightforward. For reduction of order, let $y_2 = uy_1$; taking derivatives,

$$y_2' = u'x^2 + 2ux$$
$$y_2'' = u''x^2 + 4u'x + 2u.$$

Substituting into the differential equation,

$$\begin{aligned} Ly_2 &= x^2 y_2'' + 2xy_2' - 6y_2 \\ &= x^2(u''x^2 + 4u'x + 2u) + 2x(u'x^2 + 2ux) - 6ux^2 \\ &= x^4u'' + 6x^3u' + ux^2(2 + 4 - 6) \\ &= x^4u'' + 6x^3u' = 0. \end{aligned}$$

Setting $v = u'$, we have the first order linear differential equation $xv' + 6v = 0$. The equation can be solved using the techniques of chapter 2.3. In summary, multiply by the integrating factor x^5 to obtain $(x^6v)' = 0$ with solution $v = x^{-6}$. We then obtain $u = x^{-5}$ times some constant that can be ignored. Then $y_2 = uy_1 = x^{-5}x^2 = x^{-3}$. Check that y_2 is a solution and that the set of solutions y_1, y_2 is linearly independent. Then the general solution is $y = c_1x^2 + c_2x^{-3}$. On what interval is y a solution to the differential equation?

(c) Try using the formula in chapter 4.2 to solve this equation.

2. (a) The corresponding homogeneous equation is $y'' + y' = 0$. Clearly y_1 is a solution to that equation. Reduction of order leads to the equation $u'' + u' = 0$. Letting $v = u'$ we have $v' + v = 0$ with solution $v = e^{-x}$, $u = e^{-x}$ times some constant which we can ignore, and $y_2 = uy_1 = e^{-x}$. You should check that that is a solution to the homogeneous equation and that y_1, y_2 form a fundamental set of solutions; it follows that the general solution to the homogeneous equation is $y_h = c_1 + c_2e^{-x}$.

   To use reduction of order to solve the non-homogeneous equation, consider a function $y_p(x) = w(x)y_1(x)$ where w is unknown and $y_1(x) = 1$. Substituting that into the non-homogeneous differential equation, we obtain the equation $w'' + w' = 1$. Letting $z = w'$ we have $z' + z = 1$ which can be solved using the techniques of chapter 2.3. In summary, multiply by the integrating factor e^x to obtain $(e^xz)' = e^x$. Integrating, $e^xz = e^x$ or $z = 1$. Integrating to find w, we have $w = x$ (in this case the constant factor in w does matter; we can't drop it as we have when solving homogeneous equations). Finally, we have $y_p = wy_1 = x$. You should check that that gives a particular solution to the differential equation.

   In summary, the general solution to the differential equation is $y = y_h + y_p = c_1 + c_2e^{-x} + x$.

   (b) The homogeneous equation is $y'' - 4y' + 3y = 0$. You should check that $y_1 = e^x$ is a solution. For reduction of order let $y_2 = uy_1 = ue^x$. The derivatives of y_2 are $y_2' = u'e^x + ue^x$ and $y_2'' = u''e^x + 2u'e^x + ue^x$. Substituting into the differential equation we obtain $u''e^x + 2u'e^x + ue^x - 4(u'e^x + ue^x) + 3ue^x = u''e^x - 2u'e^x + (ue^x - 4ue^x + 3ue^x) = (u'' - 2u')e^x = 0$. Letting $v = u'$, we have the differential equaiton $v' - 2v = 0$ with solution $v = e^{2x}$. Integrating to find u we have $u = e^{2x}$ times some constant which we can ignore because we are currently solving a homogeneous equation. Finally, we have $y_2 = uy_1 = e^{2x}e^x = e^{3x}$. Check that y_1 and y_2 form a fundamental system of solutions, so the general solution to the homogeneous equation is $y_h = c_1e^x + c_2e^{3x}$.

   To solve the non-homogeneous equation, do a calculation in parallel with the above: let $y_p(x) = w(x)y_1(x) = we^x$ where w is an unknown function. Taking derivatives and substituting into the non-homogeneous equation we eventually obtain $(w'' - 2w')e^x = x$. Letting $z = w'$ we obtain $z' - 2z = xe^{-x}$. Multiplying by the integrating factor e^{-2x} we obtain $e^{-2x}z' - 2e^{-2x}z = xe^{-3x}$ or $(e^{-2x}z)' = xe^{-3x}$. Integrating by parts we have $e^{-2x}z = (-x/3 - 1/9)e^{-3x}$ or $z = (-x/3 - 1/9)e^{-x}$. Integrating by parts again to find w we have $w = (x/3 + 4/9)e^{-x}$. Finally, $y_p = wy_1 = (x/3 + 4/9)e^{-x}e^x = x/3 + 4/9$ is a particular solution to the original non-homogeneous differential equation. (Check!)

   In summary, the general solution to the non-homogeneous differential equation is $y = y_h + y_p = c_1e^x + c_2e^{3x} + x/3 + 4/9$. We could have obtained the particular solution much more quickly in this case by using undetermined coefficients, but the above method gives an idea of how a particular solution could be found even when undetermined coefficients doesn't work. We will explore this idea in greater detail in chapter 4.6.

3. These equations all have constant coefficients so we use the methods of chapter 4.3.

(a) The auxiliary algebraic equation in this case is $m^2 - 3m + 2 = 0$ which immediately factors to $(m-1)(m-2) = 0$ with distinct real roots $m = 1$ and $m = 2$. Therefore a general solution is $y = c_1 e^x + c_2 e^{2x}$.

(b) The auxiliary equation is $m^2 + 4m - 1 = 0$ with solutions

$$m = \frac{-4 \pm \sqrt{16 - 4(-1)}}{2} = -2 \pm \sqrt{5}$$

again with distinct real roots, so the general solution is $y = c_1 e^{(-2+\sqrt{5})x} + c_2 e^{(-2-\sqrt{5})x}$.

(c) The auxiliary equation is $2m^2 + 2m + 1 = 0$ with solutions

$$m = \frac{-2 \pm \sqrt{4 - 4(2)(1)}}{2(2)} = -\frac{1}{2} \pm \frac{i}{2}$$

with two distinct complex roots. The general solution in this case is $y = c_1 e^{-x/2} \cos(x/2) + c_2 e^{-x/2} \sin(x/2)$. (Check!)

4. Again, these equations all have constant coefficients so we use the methods of chapter 4.3 for higher order equations.

(a) The auxiliary equation in this case is $m^3 + 3m^2 - 4m - 12 = 0$. We would have great difficulty solving the general cubic equation, so we try to guess a root. Good guesses would be the factors of 12. Neither 1 nor -1 work, so we try $m = 2$: $8 + 3(4) - 4(2) - 12 = 0$ so our guess works. Now we use the factor theorem to factor out the root $m = 2$ and obtain a quadratic equation: $(m-2)(Am^2 + Bm + C) = m^3 + 2m^2 - 4m - 12$ so $A = 1$, $B = 5$, $C = 6$ and we have $m^3 + 3m^2 - 4m - 12 = (m-2)(m^2 + 5m + 6)$. Factoring the quadratic gives $(m-2)(m+2)(m+3) = 0$ with roots $m = 2$, $m = -2$, $m = -3$. There are three distinct real roots, so the general solution is $y = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{-3x}$. (Check!)

(b) The auxiliary equation is $m^4 - 2m^2 + 1 = 0$. Again, we would have great difficulty solving a quartic equation in general, so we look instead for a trick. Let $n = m^2$, then the equation becomes $n^2 - 2n + 1 = 0$, i.e., $(n-1)^2 = 0$ with the double root $n = 1, 1$. Taking square roots we get the four roots of the original equation $m = 1, 1, -1, -1$. (Check that the original equation can be factored as $(m-1)^2(m+1)^2$.) Since there are double roots, the general solution in this case is $y = c_1 e^x + c_2 x e^x + c_3 e^{-x} + c_4 x e^{-x}$. (Check that the four functions satisfy the differential equation and are linearly independent!)

(c) The auxiliary equation in this case is $2m^5 - 7m^4 + 12m^3 + 8m^2 = 0$. Quintic equations are even further out of reach, but in this case we can simplify by taking out a factor of m^2: $m^2(2m^3 - 7m^2 + 12m + 8) = 0$. [Check back later ... something's wrong.]

5. (a) The general solution to this harmonic oscillator equation is $y = c_1 \cos 2x + c_2 \sin 2x$. Substituting the boundary condition $y(0) = 0$ we have $0 = y(0) = c_1 \cos 2(0) + c_2 \sin 2(0) = c_1$ so $c_1 = 0$. Then using the boundary condition $0 = y(\pi) = c_2 \sin 2\pi = 0$ gives no further condition on c_2. Therefore there is an infinite family of solutions to the boundary value problem of the form $y = c_2 \sin 2x$.

(b) The auxiliary equation is $m^2 - 2m + 2 = 0$ with solution

$$m = \frac{2 \pm \sqrt{4 - 4(2)}}{2} = 1 \pm i$$

so the general solution to the differential equation is $y = c_1 e^x \cos x + c_2 e^x \sin x$. (Check!) The first boundary condition gives $1 = y(0) = c_1 e^0 \cos 0 + c_2 e^0 \sin 0 = c_1$ so $c_1 = 1$ and the solution must be of the form $y = e^x \cos x + c_2 e^x \sin x$. The second boundary condition then implies $1 = y(\pi) = e^\pi \cos \pi + c_2 e^\pi \sin \pi = e^\pi(-1) + c_2 e^\pi(0) = -e^\pi$. Since 1 is not equal to $-e^\pi$ (because, e.g., 1 is positive while $-e^\pi$ is negative), the boundary conditions are contradictory and there is no solution to the given boundary value problem.