MATH281 200610 Problem Set 7 Solutions DRAFT

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1. (a) First we solve the homogeneous equation $y''+y'-6y=0$. It has auxiliary equation $m^2+m-6=0$ with solutions $m=2$ and $m=-3$, so the general solution to the homogeneous equation is $y_h = c_1e^{2x}+c_2e^{-3x}$. Next we look for a particular solution to the non-homogeneous equation of the form $y_p = Ax+B$. Substituting that into the differential equation we get

$$y_p''+y_p'-6y_p = 0+A-6(Ax+B) = 2x \implies A+B-6Ax = 2x \implies A+B=0,\ -6Ax=2x$$

so we get $A=-1/3$ and $B=1/3$. Altogether the general solution to the non-homogeneous equation is

$$y = y_h+y_p = c_1e^{2x}+c_2e^{-3x}-\frac{1}{3}x+\frac{1}{3}.$$

You should check that that is a solution.

(b) The homogeneous equation $y''-16y=0$ has general solution $y_h = c_1e^{4x}+c_2e^{-4x}$. We are going to have some difficulty because the input function $2e^{4x}$ is a particular case of the general solution to the homogeneous equation, so the obvious choice of Ae^{4x} for undetermined coefficients won't work: applying the differential operator to that function is guaranteed to give us zero. Instead we try $y_p = Axe^{4x}+Be^{4x}$. Taking derivatives we have

$$\begin{aligned} y_p' &= Ae^{4x}+4Axe^{4x}+4Be^{4x} = 4Axe^{4x}+(A+4B)e^{4x} \\ y_p'' &= 16Axe^{4x}+(5A+16B)e^{4x}. \end{aligned}$$

Substituting those expressions into the non-homogeneous differential equation we have

$$y_p''-16y_p = 16Axe^{4x}+(5A+16B)e^{4x}-16Axe^{4x}-16Be^{4x} = 5Ae^{4x} = 2e^{4x}$$

which implies $A=2/5$, B arbitrary. For simplicity we choose $B=0$. The general solution to the non-homogeneous equation is then

$$y = c_1e^{4x}+c_2e^{-4x}+\frac{2}{5}xe^{4x}.$$

You should check that solution.

(c) The homogenous equation is $y^{(4)}-y''=0$ with auxiliary equation $m^4-m^2=0$, which factors to $m^2(m-1)(m+1)=0$. The general solution to the homogeneous equation is therefore $y_h = c_1+c_2x+c_3e^x+c_4e^{-x}$. To find a particular solution, we consider each of the terms $4x$ and $2xe^{-x}$ separately and superpose the answers. For the input function $4x$ we would normally seek a particular solution of the form $y_{p_1} = Ax+B$. However, that function is in the kernel of the differential operator, so we instead look for a function of the form $y_{p_1} = Ax^3+Bx^2$. Substituting that into the non-homogeneous differential equation we obtain

$$0-(6Ax+2B) = 4x$$

which is satisfied if $A=-2/3$, $B=0$, giving a particular solution $y_{p_1} = -(2/3)x^3$. For the second input function we consider a particular solution of the form $y_{p_2} = Axe^{-x}$. Taking derivatives we have

$$
\begin{aligned}
y'_{p2} &= Ae^{-x} - Axe^{-x} \\
y''_{p2} &= -Ae^{-x} - Ae^{-x} + Axe^{-x} = -2Ae^{-x} + Axe^{x} \\
y'''_{p2} &= 3Ae^{-x} - Axe^{-x} \\
y^{(4)}_{p2} &= -4Ae^{-x} + Axe^{-x}.
\end{aligned}
$$

Substituting the above into the non-homogeneous differential equation we obtain

$$-4Ae^{-x} + Axe^{-x} + 2Ae^{-x} - Axe^{x} = -2Ae^{-x} = 2xe^{-x}$$

which is satisfied if $A = -1$, so $y_{p_2} = -xe^{-x}$. Altogether, the general solution to the original equation is

$$y = c_1 + c_2 x + c_3 e^{x} + c_4 e^{-x} - \frac{2}{3}x^3 - xe^{x}.$$

You should check that answer.

2. (a) The homogeneous equation is $2y'' + 3y' - 2y = 0$ with auxiliary equation $2m^2 + 3m - 2 = 0$. By the quadratic formula the solutions are

$$m = \frac{-3 \pm \sqrt{9 - 4(2)(-2)}}{2(2)} = \frac{-3 \pm 5}{4} = -2, \frac{1}{2}.$$

The general solution to the homogeneous equation is $y_h = c_1 e^{-2x} + c_2 e^{x/2}$. We search for a particular solution of the form $y_p = Ax + B$. Substituting into the differential equation we have

$$2y''_p + 3y'_p - 2y_p = 0 + 3(A) - 2(Ax + B) = -2Ax + (3A - 2B) = -4x - 11$$

which implies $A = 2$, $B = 17/2$. Altogether the general solution to the equation is

$$y = c_1 e^{-2x} + c_2 e^{x/2} + 4x + \frac{17}{2}.$$

Differentiating we have

$$y' = -2c_1 e^{-2x} + \frac{c_2}{2} e^{x/2} + 4$$

so the initial conditions give

$$
\begin{aligned}
y(0) &= c_1 + c_2 + \frac{17}{2} = 0 \\
y'(0) &= -2c_1 + \frac{c_2}{2} + 4 = -10.
\end{aligned}
$$

Multiplying the first equation by 2 and adding to the second we obtain $(5/2)c_2 = -33$ or $c_2 = -66/5$. Then the first equation gives $c_1 = 47/10$, so the solution to the initial value problem is

$$y = \frac{47}{10} e^{-2x} - \frac{66}{5} e^{x/2} + 2x + \frac{17}{2}.$$

You should check that that solution satisfies all conditions of the initial value problem.

(b) The general solution to the homogeneous equation is $x(t) = c_1 \cos \omega t + c_2 \sin \omega t$. To find a particular solution we must consider two cases: $\gamma \neq \pm\omega$ and $\gamma = \pm\omega$. In the first (easier) case, we search for a particular solution of the form $x_p = A \cos \gamma t + B \sin \gamma t$. Taking derivatives we have

$$
\begin{aligned}
x'_p &= -A\gamma \sin \gamma t + B\gamma \cos \gamma t \\
x''_p &= -A\gamma^2 \cos \gamma t - B\gamma^2 \sin \gamma t.
\end{aligned}
$$

Substituting into the differential equation we have

$$-A\gamma^2 \cos \gamma t - B\gamma^2 \sin \gamma t + \omega^2(A \cos \gamma t + B \sin \gamma t) = F_0 \cos \gamma t$$

which implies $A(\omega^2 - \gamma^2) = F_0$ and $B(\omega^2 - \gamma^2) = 0$. Since $\omega^2 - \gamma^2 \neq 0$ we can solve for $A = F_0/(\omega^2 - \gamma^2)$ and $B = 0$ to obtain the general solution

$$x = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{\omega^2 - \gamma^2} \cos \gamma t.$$

Differentiating,

$$x' = -c_1 \omega \sin \omega t + c_2 \omega \cos \omega t - \frac{F_0 \gamma}{\omega^2 - \gamma^2} \sin \gamma t.$$

Substituting the initial conditions,

$$\begin{aligned} x(0) &= c_1 + \frac{F_0}{\omega^2 - \gamma^2} = 0 \\ x'(0) &= c_2 = 0 \end{aligned}$$

giving $c_2 = 0$ and $c_1 = -F_0/(\omega^2 - \gamma^2)$. The solution to the initial value problem is therefore

$$x(t) = -\frac{F_0}{\omega^2 - \gamma^2} \cos \omega t + \frac{F_0}{\omega^2 - \gamma^2} \cos \gamma t.$$

You should check that the above function actually satisfied the initial value problem.
If $\gamma = \pm\omega$, the above doesn't work because $A \cos \gamma t + B \sin \gamma t$ is in the kernel of the differential operator, so we must seek another form for the particular solution. (Let's assume $\gamma = \omega$; the other case can be reduced to this case.) The usual trick in this situaion is to multiply by the independent variable, t in this case. So we seek a particular solution of the form

$$\begin{aligned} x_p &= At \cos \omega t + Bt \sin \omega t \\ x_p' &= A \cos \omega t + B \sin \omega t - A\omega t \sin \omega t + B\omega t \cos \omega t \\ x_p'' &= -A\omega \sin \omega t + B\omega \cos \omega t - A\omega \sin \omega t + B\omega \cos \omega t - A\omega^2 t \cos \omega t - B\omega^2 t \sin \omega t. \end{aligned}$$

Substituting the obve into the differential equation we obtain

$$\begin{aligned} x_p'' + \omega^2 x_p &= -A\omega^2 t \cos \omega t - B\omega^2 t \sin \omega t - 2A\omega \sin \omega t + 2B\omega \cos \omega t + A\omega^2 t \cos \omega t + B\omega^2 t \sin \omega t \\ &= -2A\omega \sin \omega t + 2B\omega \cos \omega t = F_0 \cos \omega t, \end{aligned}$$

from which we conclude that $A = 0$ and $B = F_0/2$. So in this case the solution to the differential equation is

$$\begin{aligned} x &= c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{2} t \sin \omega t \\ x' &= -c_1 \omega \sin \omega t + c_2 \omega \cos \omega t + \frac{F_0}{2} \sin \omega t + \frac{F_0 \omega}{2} t \cos \omega t. \end{aligned}$$

Substituting the initial conditions,

$$\begin{aligned} x(0) &= c_1 = 0 \\ x'(0) &= c_2 \omega = 0 \end{aligned}$$

so the solution to the initial value problem is

$$x(t) = \frac{F_0}{2} t \sin \omega t.$$

You should check that that function satisfies the conditions of the initial value problem.

The two cases $\gamma \neq \pm\omega$ and $\gamma = \pm\omega$ differ radically in a qualitative sense. In the former, the solution is bounded, and exhibits the phenomenon of 'beats' (see chapter 5 of the textbook if you are interested). In the latter case, the solution is unbounded. We call the latter case the 'resonant' case. Resonance is often seen in situations where the input function oscillates at the same frequency as the natural frequency of the system (as reflected in the homogeneous equation). Resonant systems store up energy, as if you were pushing someone on a swing. Resonance is generally something to be avoided in engineered systems, and is responsible for many interesting phenomena in natural systems, such as (possibly) the configuration of the solar system.

(c)

3. (a)

(b) The general solution to the homogenous equation is $y_h = c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x$, and a particular solution is $y_p = 2x$, giving the general solution

$$y = c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x + 2x$$
$$y' = -\sqrt{3}c_1 \sin\sqrt{3}x + \sqrt{3}c_2 \cos\sqrt{3}x + 2.$$

The boundary condition at 0 gives

$$y(0) + y'(0) = c_1 + \sqrt{3}c_2 + 2 = 0$$

(this is a boundary condition of 'Robin' type, a condition on the linear combination of the value of the function and the values of its derivatives). The boundary condition at 1 gives

$$y(1) = c_1 \cos\sqrt{3} + c_2 \sin\sqrt{3} + 2 = 0$$

4. The general solution of the differential equation on the interval $0 \leq x \leq \pi$ is $y = c_1 e^{-x}\cos 3x + c_2 e^{-x}\sin 3x + 2$. The general solution of the differential equation on the interval $x > \pi$ is $y = c_3 e^{-x}\cos 3x + c_4 e^{-x}\sin 3x$. [I forgot to include the initial conditions in the problem statement and I don't have the text handy right now. If any of you do, please let me know what the initial conditions should be.]

5. (a) Expanding the given function we have $x^3 - 5x^4$. The first term has annihilator D^4, the second has annihilator D^5, and the least common multiple of the two operators is D^5, which is therefore the constant coefficient operator of least order which annihilates the given function.

(b) The first term is annihilated by D^2, the second by $D^2 + 1$, and the third by $D^2 + 25$. Therefore an operator which annihilates the sum of functions is $D^2(D^2 + 1)(D^2 + 25)$. That operator has the lowest order unless we are allowed to use complex coefficients.

(c) Expanding, the function of interest is $4 - 4e^x + e^{2x}$. The first term is annihilated by D, the second by $D - 1$, and the third by $D - 2$ so an operator which annihilates the sum of funtions is $D(D-1)(D-2)$.

(d) The first function is annihilated by an operator with $-1 + i$ as a root to its auxiliary equations; since root come in conjugate pairs for operators with real coefficients we know the other root would have to be $-1 - i$ so an operator which annihiltes the sin term would be $(D - (-1 + i))(D - (-1 - i)) = D^2 + 2D - 2$. Similarly, an annihilator for $e^{2x}\cos x$ is $D - (2 + i)$; to make it an operator with real coefficients we must multiply by its complex conjugate to obtain $(D - (2+i))(D - (2-i)) = D^2 - 4D + 5$. Altogether, the simplest real-coefficient operator which annihilates the sum is $(D^2 + 2D - 2)(D^2 - 4D + 5)$.

6. This question essentially calls for the solution to a constant-coefficient homogeneous equation.

(a) The order is 2. The operator factors as $D(D+4)$. Two linearly independent functions annihilated by the operator are 1 and e^{-4x}.

(b) The order is 2. The operator factors as $(D+3)(D-12)$. Two linearly independent functions annihilated by the operator are e^{-3x} and e^{12x}.

(c) The order is 2. The operator does not factor over the real numbers, but the roots of the auxiliary equation are

$$m = \frac{6 \pm \sqrt{36-40}}{2} = 3 \pm i.$$

Two linearly indepedent functions annihilated by the operator are $e^{3x}\cos x$ and $e^{3x}\sin x$.

(d) The order of the operator is 4. The operator is already factored. Four linearly independent functions annihilated byt the operator are 1, x, e^{5x} and e^{7x}.

7. (a) The general solution to the homogeneous equation is $y = c_1 + c_4 e^{-3x}$. An annihilator for the input function is D^2, so any function which satisfies the given differential equation is also in the kernel of $D^2(D^2+3D) = D^3(D+3)$. Such functions are of the form $y_a = c_1 + c_2 x + c_3 x^2 + c_4 e^{-3x}$. The c_1 and $c_4 e^{-3x}$ terms cannot contribute to the non-homogeneous equation, so the solution to the non-homogeneous equation must be of the form $y_p = c_2 x + c_3 x^2$. Substituting that function into the differential equation we have $2c_3 + 3c_2 + 6c_3 x = 4x - 5$ which implies $c_3 = 2/3$ and $c_2 = -19/6$. You should check that $y = c_1 + c_4 e^{-3x} - (19/6)x + (2/3)x^2$ is a general solution to the equation.

(b) An annihilator for the input function is $D^2(D-1)^2$, so any function which satifies the differential equation must also be in the kernel of $D^2(D-1)^2(D^2+3D-10) = D^2(D-1)^2(D-2)(D+5)$. Any such function is of the form $y = c_1 + c_2 x + c_3 e^x + c_4 x e^x + c_5 e^{2x} + c_6 e^{-5x}$. The latter two terms cannot contribute anything to a particular solution to the original differential equation because they are in the kernel of the original operator, so we should look for a solution of the form $y_p = c_1 + c_2 x + c_3 e^x + c_4 x e^x$. Taking derivatives,

$$\begin{aligned} y_p' &= c_2 + (c_3 + c_4)e^x + c_4 x e^x \\ y_p'' &= (c_3 + 2c_4)e^x + c_4 x e^x \end{aligned}$$

and substituting those into the original differential equation we have

$$(c_3+2c_4)e^x + c_4 x e^x + 3(c_2 + (c_3+c_4)e^x + c_4 x e^x) - 10(c_1 + c_2 x + c_3 e^x + c_4 x e^x) = (3c_2 - 10c_1) - 10c_2 x + (-7c_3 + 5c_4)e^x - 6c_4 x$$

which implies

$$\begin{aligned} 3c_2 - 10c_1 &= 0 \\ -10c_2 &= 1 \\ -7c_3 + 5c_4 &= 0 \\ -6c_4 &= 1. \end{aligned}$$

Solving, $c_4 = -1/6$, $c_3 = -5/42$, $c_2 = -1/10$ and $c_1 = -3/100$. You should check that the general solution to the given equation is

$$y = -\frac{3}{100} - \frac{1}{10}x - \frac{5}{42}e^x - \frac{1}{6}x e^x + c_5 e^{2x} + c_6 e^{-5x}.$$

(c) An annihilator for the input fuction is $(D+1)^3$, so any function which satisfies the differential equation is in the kernel of $(D+1)^3(D^2+2D+1) = (D+1)^5$. Such functions are of the form

$$y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4)e^{-x}.$$

The c_1 and c_2 terms can't contribute to the solution to the original equation because they are in the kernel of the original operator, so we look for a solution to the original equation of the form

$$y_p = (c_3 x^2 + c_4 x^3 + c_5 x^4)e^{-x}.$$

Taking derivatives,

$$y_p' = (2c_3x + 3c_4x^2 + 4c_5x^3 - c_3x^2 - c_4x^3 - c_5x^4)e^{-x} = (2c_3x + (3c_4 - c_3)x^2 + (4c_5 - c_4)x^3 - c_5x^4)e^{-x}$$
$$y_p'' = (2c_3 + (6c_4 - 4c_3)x + (12c_5 - 6c_4 + c_3)x^2 + (-8c_5 + c_4)x^3 + c_5x^4)e^{-x}.$$

Substituting into the original differential equation we have

$$(2c_3+(6c_4-4c_3)x+(12c_5-6c_4+c_3)x^2+(-8c_5+c_4)x^3+c_5x^4)+2(2c_3x+(3c_4-c_3)x^2+(4c_5-c_4)x^3-c_5x^4)+(c_3x^2+c_4x^3+$$

which implies $c_3 = 0$, $c_4 = 0$, and $12c_5 = 1$. Note that x^3 and x^4 terms vanish automatically. This gives us a general solution to the original equation

$$y = c_1e^{-x} + c_2xe^{-x} + \frac{1}{12}x^4e^{-x}.$$

You should check that the above function really is a general solution to the equation.