MATH281 200610 Problem Set 8 Solutions DRAFT

Edward Doolittle

Wednesday, March 15, 2006

1. The Wronskian of the fundamental system of solutions $\{\cos\theta, \sin\theta\}$ is

$$W = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta - -\sin^2\theta = \cos^2\theta + \sin^2\theta = 1.$$

   (a) In this case we have

$$W_1 = \begin{vmatrix} 0 & \sin\theta \\ \tan\theta & \cos\theta \end{vmatrix} = -\frac{\sin^2\theta}{\cos\theta}, \quad W_2 = \begin{vmatrix} \cos\theta & 0 \\ -\sin\theta & \tan\theta \end{vmatrix} = \sin\theta.$$

   The varied parameters are therefore

$$\begin{aligned} c_1(\theta) &= \int \frac{W_1}{W}\,d\theta = \int \cos\theta - \sec\theta\,d\theta = \sin\theta - \ln|\sec\theta + \tan\theta| \\ c_2(\theta) &= \int \frac{W_2}{W}\,d\theta = \int \sin\theta\,d\theta = -\cos\theta, \end{aligned}$$

   so the general solution to the equation is

$$\begin{aligned} y &= c_1\cos\theta + c_2\sin\theta + (\sin\theta - \ln|\sec\theta + \tan\theta|)\cos\theta - \cos\theta\sin\theta \\ &= c_1\cos\theta + c_2\sin\theta - \ln|\sec\theta + \tan\theta|\cos\theta. \end{aligned}$$

   You should check that the above solution is correct.

   (b) In this case

$$W_1 = \begin{vmatrix} 0 & \sin\theta \\ \sec\theta\tan\theta & \cos\theta \end{vmatrix} = -\tan^2\theta, \quad W_2 = \begin{vmatrix} \cos\theta & 0 \\ -\sin\theta & \sec\theta\tan\theta \end{vmatrix} = \tan\theta.$$

   Then

$$\begin{aligned} c_1(\theta) &= \int -\tan^2\theta\,d\theta = \theta - \tan\theta \\ c_2(\theta) &= \int \tan\theta\,d\theta = \ln|\sec\theta| \end{aligned}$$

   and a general solution to the equation is

$$y = c_1\cos\theta + c_2\sin\theta + (\theta - \tan\theta)\cos\theta + \ln|\sec\theta|\sin\theta.$$

   You should check the above answer.

   (c) In this case

$$W_1 = \begin{vmatrix} 0 & \sin\theta \\ \sec^2\theta & \cos\theta \end{vmatrix} = -\frac{\sin\theta}{\cos^2\theta}, \quad W_2 = \begin{vmatrix} \cos\theta & 0 \\ -\sin\theta & \sec^2\theta \end{vmatrix} = \sec\theta.$$

Then

$$c_1(\theta) = \int -\frac{\sin\theta}{\cos^2\theta}\,d\theta = -\sec\theta$$
$$c_2(\theta) = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|$$

so the general solution to the equation is

$$y = c_1\cos\theta + c_2\sin\theta - 1 + \ln|\sec\theta + \tan\theta|\sin\theta.$$

As usual, you should check the answer.

2. The homogeneous equation in the question is $y'' - 2y' + y = 0$ with general solution $y_h = c_1e^t + c_2te^t$. The Wronskian is

$$W = \begin{vmatrix} e^t & te^t \\ e^t & te^t + e^t \end{vmatrix} = te^{2t} + e^{2t} - te^{2t} = e^{2t}$$

(a) Here

$$W_1 = \begin{vmatrix} 0 & te^t \\ e^t(1+t^2)^{-1} & te^t + e^t \end{vmatrix} = \frac{te^{2t}}{1+t^2}, \quad W_2 = \begin{vmatrix} e^t & 0 \\ e^t & e^t(1+t^2)^{-1} \end{vmatrix} = \frac{e^{2t}}{1+t^2}.$$

Therefore

$$c_1(t) = \int \frac{t}{1+t^2}\,dt = \frac{1}{2}\ln(1+t^2)$$
$$c_2(t) = \int \frac{1}{1+t^2}\,dt = \arctan(t)$$

so the general solution to the equation is

$$y = c_1e^t + c_2te^t + \frac{1}{2}\ln(1+t^2)e^t + \arctan(t)te^t.$$

You should check that solution.

(b) In this case

$$W_1 = \begin{vmatrix} 0 & te^t \\ e^t\arctan t & te^t + e^t \end{vmatrix} = -e^{2t}t\arctan t, \quad W_2 = \begin{vmatrix} e^t & 0 \\ e^t & e^t\arctan t \end{vmatrix} = e^{2t}\arctan t.$$

Integrating by parts we have

$$c_1(t) = \int -t\arctan t\,dt = -\frac{t^2}{2}\arctan t + \frac{1}{2}\int\frac{t^2}{1+t^2}\,dt = -\frac{t^2}{2}\arctan t + \frac{1}{2}(t - \arctan t)$$
$$c_2(t) = \int \arctan t\,dt = t\arctan t - \int\frac{t}{1+t^2}\,dt = t\arctan t - \frac{1}{2}\ln(1+t^2)$$

so the general solution to the equation is

$$y = c_1e^t + c_2te^t + \frac{1}{2}(t - \arctan t - t^2\arctan t)e^t + \frac{1}{2}(2t\arctan t - \ln(1+t^2))te^t.$$

As usual, you should check the solution.

(c)

$$W_1 = \begin{vmatrix} 0 & te^t \\ e^t\ln t & te^t + e^t \end{vmatrix} = -e^{2t}t\ln t, \quad W_2 = \begin{vmatrix} e^t & 0 \\ e^t & e^t\ln t \end{vmatrix} = e^{2t}\ln t.$$

Integrating by parts we have

$$c_1(t) = \int -t\ln t\, dt = -\frac{t^2}{2}\ln t + \int \frac{t}{2}\, dt = \frac{t^2}{4} - \frac{t^2}{2}\ln t$$
$$c_2(t) = \int \ln t\, dt = t\ln t - \int 1\, dt = t\ln t - t$$

giving the general solution

$$y = c_1 e^t + c_2 te^t + \frac{1}{4}(t^2 - 2t^2\ln t)e^t + (t\ln t - t)te^t.$$

You know what I'm going to suggest at this point.

3. (a)
   (b)
   (c)
4. (a)
   (b)
5. The homogeneous equation is $y''' + 4y' = 0$ with auxiliary equation $m^3 + 4m = 0$ which has roots $m = 0, +2i, -2i$. Therefore the homogeneous equation has fundamental system of solutions $1, \cos 2x, \sin 2x$. The Wronskian is

$$W = \begin{vmatrix} 1 & \cos 2x & \sin 2x \\ 0 & -2\sin 2x & 2\cos 2x \\ 0 & -4\cos 2x & -4\sin 2x \end{vmatrix} = 8\sin^2 2x + 8\cos^2 2x = 8$$

and the other determinants are

$$W_1 = \begin{vmatrix} 0 & \cos 2x & \sin 2x \\ 0 & -2\sin 2x & 2\cos 2x \\ \sec 2x & -4\cos 2x & -4\sin 2x \end{vmatrix} = \sec 2x(2\cos^2 2x + 2\sin^2 2x) = \sec 2x$$
$$W_2 = \begin{vmatrix} 1 & 0 & \sin 2x \\ 0 & 0 & 2\cos 2x \\ 0 & \sec 2x & -4\sin 2x \end{vmatrix} = 1(-2\cos 2x \sec 2x) = -2$$
$$W_3 = \begin{vmatrix} 1 & \cos 2x & 0 \\ 0 & -2\sin 2x & 0 \\ 0 & -4\cos 2x & \sec 2x \end{vmatrix} = 1(-2\sin 2x \sec 2x) = -2\tan 2x.$$

Therefore

$$c_1(x) = \int \frac{\sec 2x}{8}\, dx = \frac{1}{16}\ln|\sec 2x + \tan 2x|$$
$$c_2(x) = \int \frac{-2}{8}\, dx = -\frac{1}{4}x$$
$$c_3(x) = \int \frac{-2\tan 2x}{8}\, dx = -\frac{1}{8}\ln|\sec 2x|$$

and the general solution to the non-homogeneous equation is

$$y = c_1 + c_2\cos 2x + c_3\sin 2x + \frac{1}{16}\ln|\sec 2x + \tan 2x| - \frac{1}{4}x\cos 2x - \frac{1}{8}\ln|\sec 2x|\sin 2x.$$

You should check the above.

6. (a) This question was posed on an earlier problem set.
   (b) This question was posed on an earlier problem set.
   (c) The Wronskian is $W = 1/x$ and we have

$$W_1 = \begin{vmatrix} 0 & \sin(\ln x) \\ \sec(\ln x) & \frac{1}{x}\cos(\ln x) \end{vmatrix} = \tan(\ln x)$$
$$W_2 = \begin{vmatrix} \cos(\ln x) & 0 \\ -\frac{1}{x}\sin(\ln x) & \sec(\ln x) \end{vmatrix} = 1.$$

Therefore (there seems to be a problem below)

$$c_1(x) = \int \frac{\tan(\ln x)}{1/x}\, dx = \int e^{2u} \tan u \, du c_2(x) \qquad = \int \frac{1}{1/x}\, dx = \int x\, dx = \frac{x^2}{2}$$

and a general solution to the differential equation is

$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x) + () \cos(\ln x) + \frac{x^2}{2} \sin(\ln x).$$

You should check your answer.

7. The homogeneous equation is $y'' - 2y' + y = 0$ with fundamental set of solutions e^x, xe^x. A particular solution to the equation $y'' - 2y' + y = 4x^2 - 3$ can be found by undetermined coefficients. Let

$$y_{p_1} = Ax^2 + Bx + C$$
$$y'_{p_1} = 2Ax + B$$
$$y''_{p_1} = 2A$$

so

$$y''_{p_1} - 2y'_{p_1} + y_{p_1} = 2A - 2(2Ax + B) + Ax^2 + Bx + C = Ax^2 + (B - 4A)x + (C - 2B + 2A) = 4x^2 - 3$$

which implies $A = 4$, $B = 1$, $C = -9$, so

$$y_{p_1} = 4x^2 + x - 9.$$

A particular solution to $y'' - 2y' + y = x^{-1}e^x$ cannot be found by undetermined coefficients so we must use variation of parameters. The Wronskian of the fundamental set of solutions is $W = e^{2x}$ and we have

$$W_1 = \begin{vmatrix} 0 & xe^x \\ x^{-1}e^x & xe^x + e^x \end{vmatrix} = e^{2x}, \quad W_2 = \begin{vmatrix} e^x & 0 \\ e^x & x^{-1}e^x \end{vmatrix} = x^{-1}e^{2x}.$$

Therefore

$$c_1(x) = \int 1\, dx = x$$
$$c_2(x) = \int x^{-1}\, dx = \ln x$$

and a particular solution to $y'' - 2y' + y = x^{-1}e^x$ is

$$y_{p_2} = xe^x + (\ln x)xe^x.$$

By the principle of superposition

$$y_p = y_{p_1} + y_{p_2} = 4x^2 + x - 9 + xe^x + (\ln x)xe^x$$

is a particular solution to the equation $y'' - 2y' + y = 4x^2 - 3 + x^{-1}e^x$, so a general solution to the equation is

$$y = c_1 e^x + c_2 xe^x + 4x^2 + x - 9 + (\ln x)xe^x.$$

You should check the above solution.

8. (a) Substituting $y = x^m$ into the equation we have

$$m(m-1)x^{m+2} + mx^{m+2} - 4x^{m+2} = 0.$$

Dividing through by x^{m+2} gives the auxiliary equation

$$m(m-1) + m - 4 = 0 \implies m^2 - 4 = 0$$

with one solution $m = \pm 2$. Therefore $y = x^2$ a solution to the differential equation. (Check!)

(b) The other root of the auxiliary equation is $m = -2$ which gives the fundamental set of solutions x^2, x^{-2}. (The second solution could also be found by reduction of order.) The solutions are linearly independent because the Wronskian $W = x^2(-2)x^{-3} - x^{-2}(2x) = -4x^{-1}$ is never 0.

(c) The other determinants are

$$W_1 = \begin{vmatrix} 0 & x^{-2} \\ 1 & -2x^{-3} \end{vmatrix} = x^{-2}, \quad W_2 = \begin{vmatrix} x^2 & 0 \\ 2x & 1 \end{vmatrix} = x^2$$

so

$$c_1(x) = \int \frac{x^{-2}}{-4x^{-1}}\,dx = -\frac{1}{4}\int \frac{1}{x}\,dx = -\frac{1}{4}\ln x$$
$$c_2(x) = \int \frac{x^2}{-4x^{-1}}\,dx = -\frac{1}{4}\int x^3\,dx = -\frac{1}{16}x^4$$

and the general solution to the equation is

$$y = c_1x^2 + c_2x^{-2} - \frac{1}{4}x^2\ln x - \frac{1}{16}x^2.$$

Check.