MATH281 200610 Problem Set 9 Solutions DRAFT

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1. Let S be the sum under consideration. The first and third series are already of the appropriate form. For the middle series, make the change of variable $k = n - 2$, $n = k + 2$ to obtain

$$S = \sum_{k=2}^{\infty} k(k-1)c_k x^k + 2\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + 3\sum_{k=1}^{\infty} kc_k x^k.$$

Now the difficulty is in dealing with the varying lower bounds for k. Sometimes the trick of setting $c_{-1}, c_{-2}, \ldots$ equal to zero works, but not in this case. (Why not?) So instead, we separate the terms which are in each of the above series from those which are exceptional.

$$\begin{aligned} S &= \sum_{k=2}^{\infty} k(k-1)c_k x^k + 2(0+2)(0+1)c_{0+2}x^0 + 2(1+2)(1+1)c_{1+2}x^1 + 2\sum_{k=2}^{\infty}(k+2)(k+1)c_{k+2}x^k \\ &\quad + 3(1)c_1 x^1 + 3\sum_{k=2}^{\infty} kc_k x^k \\ &= 4c_2 + (12c_3 + 3c_1)x + \sum_{k=2}^{\infty}\left(k(k-1)c_k + 2(k+2)(k+1)c_{k+2} + 3kc_k\right)x^k \\ &= 4c_2 + (12c_3 + 3c_1)x + \sum_{k=2}^{\infty}\left((k^2+2k)c_k + (k+2)(k+1)c_{k+2}\right)x^k. \end{aligned}$$

2. Differentiating,

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} 2nx^{2n-1} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n-1}n!(n-1)!} x^{2n-1} \\ y'' &= \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n-1}n!(n-1)!}(2n-1)x^{2n-2}. \end{aligned}$$

(Why do both sums start at $n = 1$?) Multiplying by the coefficients,

$$\begin{aligned} xy &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n+1} \\ y' &= \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n-1}n!(n-1)!} x^{2n-1} \\ xy'' &= \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n-1}n!(n-1)!}(2n-1)x^{2n-1}. \end{aligned}$$

Making the change of index $k = n + 1$, $n = k - 1$ in the series for xy we have

$$\begin{aligned} xy &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2^{2k-2}((k-1)!)^2} x^{2k-1} \\ y' &= \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k-1}k!(k-1)!} x^{2n-1} \\ xy'' &= \sum_{n=1}^{\infty} \frac{(-1)^k}{2^{2k-1}k!(k-1)!}(2k-1)x^{2k-1}. \end{aligned}$$

Adding,

$$\begin{aligned} Ly &= xy'' + y' + xy \\ &= \sum_{k=1}^{\infty} \left(\frac{(-1)^{k-1}}{2^{2k-2}((k-1)!)^2} + \frac{(-1)^k}{2^{2k-1}k!(k-1)!} + \frac{(-1)^k}{2^{2k-1}k!(k-1)!} \right) x^{2k-1} \\ &= \sum_{k=1}^{\infty} (-1)^{k-1} \left(\frac{2^2k^2}{2^{2k}(k!)^2} - \frac{2k}{2^{2k}(k!)^2} - \frac{2k(2k-1)}{2^{2k}(k!)^2} \right) x^{2k-1} \\ &= \sum_{k=1}^{\infty} (-1)^{k-1} 0 x^{2k-1} = 0 \end{aligned}$$

as required.

3. In each case, we let the power series be $y = \sum_{n=0}^{\infty} c_n x^n$. Then, differentiating,

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n c_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}. \end{aligned}$$

(a) Multiplying by the coefficients,

$$x^2 y = \sum_{n=0}^{\infty} c_n x^{n+2}.$$

Then

$$Ly = y'' + x^2 y = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+2}.$$

In order to get the series "in sync" we shift the second series by the change of variables $k-2 = n+2$, $k = n + 4$, $n = k - 4$:

$$Ly = \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} + \sum_{k=4}^{\infty} c_{k-4} x^{k-2}.$$

Adding corresponding terms, and treating the powers of x that don't appear in both series as exceptional,

$$Ly = 2c_2 + 6c_3 x + \sum_{k=4}^{\infty} \left(k(k-1)c_k + c_{k-4} \right) x^{k-2}.$$

In order for the series to be identically zero we have c_0 and c_1 arbitrary, $c_2 = c_3 = 0$, and $c_k = -c_{k-4}/(k(k-1))$ for $k = 4, 5, \ldots$. We can find two linearly independent power series solutions by setting $c_0 = 1$, $c_1 = 0$ and $c_0 = 0$, $c_1 = 1$ respectively. In the first case we have

$$c_0 = 1, \quad c_4 = -\frac{1}{4(3)}, \quad c_8 = \frac{1}{8(7)(4)(3)}, \ldots,$$

with all other coefficients zero; in the second case we have

$$c_1 = 1, \quad c_5 = -\frac{1}{5(4)}, \quad c_9 = \frac{1}{9(8)(5)(4)}, \ldots,$$

with all other coefficients zero.

(b) Multiplying the series for y, y' and y'' by the appropriate coefficients we have

$$\begin{aligned} 2y &= \sum_{n=0}^{\infty} 2c_n x^n \\ 2xy' &= \sum_{n=1}^{\infty} 2nc_n x^n \\ y'' &= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \end{aligned}$$

We need to shift the series so they have the terms have the same power of x. We can either change them all to x^k or change them all to x^{k-2}. Either way works, but let's follow the textbook and systematically change them all to x^k. The first and second series stay the same, while in the third we make the change of variable $k = n-2$, $n = k+2$:

$$\begin{aligned} 2y &= \sum_{k=0}^{\infty} 2c_k x^k \\ 2xy' &= \sum_{k=1}^{\infty} 2kc_k x^k \\ y'' &= \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k. \end{aligned}$$

We could treat the terms with $k = 0$ in the first and third series as exceptional, but in this case we have a trick at our disposal. We can write

$$\begin{aligned} 2y &= \sum_{k=0}^{\infty} 2c_k x^k \\ 2xy' &= \sum_{k=0}^{\infty} 2kc_k x^k \\ y'' &= \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k \end{aligned}$$

where the second series can be taken from $k = 0$ because the coefficient is 0 when $k = 0$. Adding, we have

$$\begin{aligned} Ly &= y'' + 2xy' + 2y \\ &= \sum_{k=0}^{\infty} \left(2(k+1)c_k + (k+1)(k+2)c_{k+2}\right) x^k. \end{aligned}$$

That gives the following condition for a solution: $2(k+1)c_k = -(k+1)(k+2)c_{k+2}$. Dividing through by $k+1$ (which is never 0: why?), we have $c_{k+2} = -2c_k/(k+2)$. The coefficients c_0 and c_1 are arbitrary, so for two linearly independent solutions we set $c_0 = 1$, $c_1 = 0$ and $c_0 = 0$, $c_1 = 1$ respectively. In the first case we have

$$c_0 = 1, \quad c_2 = -2/2 = -1, \quad , c_4 = -2c_2/4 = 1/2, \quad c_6 = -2c_4/6 = -1/6, \quad , \ldots,$$

and in general

$$c_{2k} = \frac{(-2)^k}{(2k)(2k-2)\cdots(2)}, \ldots,$$

with all other coefficients zero. In the second case we have

$$c_{2k+1} = \frac{(-2)^k}{(2k+1)(2k-1)\cdots(3)(1)}$$

with all other coefficients zero.

(c) Three term recurrence relation.

4. Find a series solution for the differential equation as in the previous problem, and then use the initial conditions to find the values of c_0 and c_1.

   (a) In this case $c_0 = 2$, $c_1 = -1$.

   (b) In this case $c_0 = 0$, $c_1 = 1$.

5. (a) By the definition of the Laplace transform,

$$\mathscr{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^2 e^{-st} 4\, dt = -\frac{4}{s}e^{-2s} - -\frac{4}{s}e^{-0s} = \frac{4-4e^{-2s}}{s}$$

where the interval of integration was changed from $(0,\infty)$ to $(0,2)$ because f vanishes outside of $(0,2)$.

   (b) By the same reasoning as in the previous problem,

$$\mathscr{L}\{f\}(s) = \int_0^1 e^{-st}(2t+1)\, dt = 2\int_0^1 e^{-st} t\, dt + 2\int_0^1 e^{-st}\, dt.$$

Integrating the first integral by parts with $u = t$, $dv = e^{-st}\, dt$,

$$\mathscr{L}\{f\}(s) = -\frac{2}{s}te^{-st}\Big|_0^1 + \int_0^1 \frac{2}{s}e^{-st}\, dt + -\frac{2}{s}e^{-st}\Big|_0^1 = -\frac{2}{s}e^{-s} - \frac{2}{s^2}e^{-s} + \frac{2}{s^2} - \frac{2}{s}e^{-s} + \frac{2}{s}.$$

6. (a) By the definition of the Laplace transform,

$$\mathscr{L}\{e^{-2t-5}\}(s) = \int_0^\infty e^{-st}e^{-2t-5}\, dt = e^{-5}\int_0^\infty e^{-(s+2)t}\, dt = -\frac{e^{-5}}{s+2}e^{-(s+2)t}\Big|_0^\infty = \frac{e^{-5}}{s+2}.$$

The same result could have been obtained by linearity and a simple table of Laplace transforms.

   (b) Integrate by parts twice and solve for the Laplace transform.

   (c) Integrate by parts once.

7. (a) By linearity we have

$$\mathscr{L}\{f\}(s) = -4\mathscr{L}\{t^2\}(s) + 16\mathscr{L}\{t\}(s) + 9\mathscr{L}\{1\} = -4\frac{2}{s^3} + 16\frac{1}{s^2} - 9\frac{1}{s}.$$

   (b) We need to expand the polynomial before we can take the Laplace transform. By the binomial theorem,

$$f(t) = 8t^3 - 12t^2 + 6t - 1$$

so

$$\mathscr{L}\{f\}(s) = \frac{48}{s^4} - \frac{24}{s^3} + \frac{6}{s^2} - \frac{1}{s}.$$

(c) Again, we don't know how to take Laplace transforms of squares of functions, so we have to expand the square using the binomial theorem:

$$f(t) = e^{2t} - 2 + e^{-2t}.$$

Taking the Laplace transform, applying the linear property, and employing a table of Laplace transforms, we have

$$\mathscr{L}\{f\}(s) = \frac{1}{s-2} - \frac{2}{s} + \frac{1}{s+2}.$$

8. (a) By the double angle formula, $\cos^2 t = (1 + \cos 2t)/2$ so the Laplace transform is

$$\mathscr{L}\{\cos^2 t\}(s) = \frac{1}{2}\mathscr{L}\{1\}(s) + \frac{1}{2}\mathscr{L}\{\cos 2t\}(s) = \frac{1}{2s} + \frac{s}{2s^2+8}.$$

(b) By the angle addition formula for cos we have

$$f(t) = 10\cos t \cos\frac{\pi}{6} + 10\sin t \sin\frac{\pi}{6} = 5\cos t + 5\sqrt{3}\sin t$$

so

$$\mathscr{L}\{f\}(s) = \frac{5s}{s^2+1} + \frac{5\sqrt{3}}{s^2+1} = \frac{5s+5\sqrt{3}}{s^2+1}.$$