MATH281 200610 Problem Set 10 Solutions DRAFT

Edward Doolittle

Wednesday, April 5, 2006

1. (a) Expanding the square,

$$\left(\frac{2}{s}-\frac{1}{s^3}\right)^2=\frac{4}{s^2}-\frac{4}{s^4}+\frac{1}{s^6}.$$

By the linearity of the inverse Laplace transform, the answer is

$$\mathscr{L}^{-1}\left\{\frac{4}{s^2}-\frac{4}{s^4}+\frac{1}{s^6}.\right\}=4\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\}-\frac{4}{6}\mathscr{L}^{-1}\left\{\frac{3!}{s^4}\right\}+\frac{1}{120}\mathscr{L}^{-1}\left\{\frac{5!}{s^6}\right\}=4t-\frac{2}{3}t^3+\frac{1}{120}t^5.$$

(b)

(c)

2. (a) The denominator factors as $s^2+s-20=(s+5)(s-4)$. By partial fractions the argument of the Laplace transform is

$$\frac{1}{(s+5)(s-4)}=\frac{A}{s+5}+\frac{B}{s-4}=\frac{A(s-4)+B(s+5)}{(s+5)(s-4)}.$$

Expanding the numerator on the right hand side and comparing terms with the numerator on the left hand side, we have

$$(A+B)s+(-4A+5B)=0s+1,$$

leading to the system of equations

$$\begin{aligned}A+B&=0\\-4A+5B&=1\end{aligned}$$

with solution $A=-1/9$, $B=1/9$. Therefore we have

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+s-20}\right\}=-\frac{1}{9}\mathscr{L}^{-1}\left\{\frac{1}{s+5}\right\}+\frac{1}{9}\mathscr{L}^{-1}\left\{\frac{1}{s-4}\right\}=-\frac{1}{9}e^{-5t}+\frac{1}{9}e^{4t}.$$

You should check by taking the Laplace transform of the above function.

(b) We need to factor the numerator as far as possible before proceeding. Note that s^4-9 is a difference of squares, so we have $s^4-9=(s^2-3)(s^2+3)$. The latter factor cannot be factored further but the former is also a difference of squares, so altogether we have

$$s^4-9=(s-\sqrt{3})(s+\sqrt{3})(s^2+3).$$

Now by partial fractions,

$$\frac{1}{s^4-9}=\frac{A}{s-\sqrt{3}}+\frac{B}{s+\sqrt{3}}+\frac{Cs+D}{s^2+3}=\frac{A(s+\sqrt{3})(s^2+3)+B(s-\sqrt{3})(s^2+3)+(Cs+D)(s^2-3)}{s^4-9}.$$

We could further expand the numerator and compare with $0s^3+0s^2+0s+1$, but it is faster to use the trick of substituting certain special values of s. First, letting $s=\sqrt{3}$, we have $1=B\cdot 2\sqrt{3}(6)$

or $B = 1/(12\sqrt{3})$. Next, letting $s = -\sqrt{3}$, we have $A = -1/(12\sqrt{3})$. If you're comfortable with complex numbers you can get C and D by letting $s = \sqrt{3}i$; otherwise you can set $s = 0$ to obtain $1 = (1/12) + (1/12) + D(3)$, i.e., $D = -1/18$, and then comparing the s^3 term on each side we have $A + B + C = 0$ which implies $C = 0$. At this point you should double check that the partial fractions decomposition is correct. Now the inverse Laplace transform is easy:

$$\mathscr{L}^{-1}\left\{\frac{1}{s^4-9}\right\} = -\frac{1}{12\sqrt{3}}e^{\sqrt{3}t} + \frac{1}{12\sqrt{3}}e^{-\sqrt{3}t} - \frac{1}{54}\sin 3t.$$

You should check by taking the Laplace transform of the above function. You could have saved some work by using the hyperbolic trig functions if you are comfortable with that.

(c) The denominator is already factored, so the partial fractions decomposition is of the form

$$\frac{s^2+1}{s(s-1)(s+1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1} + \frac{D}{s-2}.$$

Clearing fractions and comparing numerators, we must have

$$s^2 + 1 = A(s-1)(s+1)(s-2) + Bs(s+1)(s-2) + Cs(s-1)(s-2) + Ds(s-1)(s+1).$$

Since the denominator has distinct real roots, this problem is an ideal candidate for the substitution trick. Setting $s = 0$, $s = 1$, $s = -1$, and $s = 2$ successively gives $A = 1/2$, $B = -1$, $C = -1/3$, and $D = -5/6$. At this point you should check the partial fraction decomposition. It follows that the inverse Laplace transform is

$$\mathscr{L}^{-1}\left\{\frac{s^2+1}{s(s-1)(s+1)(s-2)}\right\} = \frac{1}{2} - e^t - \frac{1}{3}e^{-t} - \frac{5}{6}e^{2t}.$$

You should check the answer by finding the Laplace transform of the above function.

3. By convention, we write $\mathscr{L}y(t) = Y(s)$ in the following.

(a) Taking the Laplace transform of both sides of the equation,

$$sY(s) - y(0) - Y(s) = \frac{2s}{s^2+25}.$$

Using the initial condition $y(0) = 0$ and solving for $Y(s)$, we have

$$Y(s) = \frac{2s}{(s-1)(s^2+25)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+25} = \frac{A(s^2+25) + (Bs+C)(s-1)}{(s-1)(s^2+25)}.$$

Setting $s = 1$ gives $A = 1/13$. Setting $s = 5i$ gives

$$(5iB + C)(5i - 1) = 10i \implies -25B - 5iB + 5iC - C = 10i \implies -25B - C = 0 \text{ and } -B + C = 2$$

from which we get $B = -1/13$ and $C = 27/13$. (Alternatively, setting $s = 0$ helps in this case.) Now, taking the inverse Laplace transform,

$$y(t) = \frac{1}{13}e^t - \frac{1}{13}\cos 5t + \frac{27}{65}\sin 5t.$$

You should check that the above function satisfies the initial value problem.

(b) Taking the Laplace transform of both sides,

$$s^2Y(s) - sy(0) - y'(0) - 4sY(s) + 4y(0) = \frac{6}{s-3} - \frac{3}{s+1}.$$

Using the initial conditions and solving for $Y(s)$,

$$\begin{aligned}(s^2-4s)Y(s)-s-3 &= \frac{6}{s-3}-\frac{3}{s+1}\\ Y(s) &= \frac{6}{s(s-4)(s-3)}-\frac{3}{s(s-4)(s+1)}+\frac{s+3}{s(s-4)}.\end{aligned}$$

Applying partial fractions decomposition to the first term,

$$\frac{6}{s(s-4)(s-3)}=\frac{A}{s}+\frac{B}{s-4}+\frac{C}{s-3}=\frac{A(s-4)(s-3)+Bs(s-3)+Cs(s-4)}{s(s-4)(s-3)}.$$

Setting $s=0$, $s=4$, $s=3$ successively gives $A=1/2$, $B=3/2$, $C=-2$ (check). The second term is

$$\frac{3}{s(s-4)(s+1)}=\frac{D}{s}+\frac{E}{s-4}+\frac{F}{s+1}=\frac{D(s-4)(s+1)+Es(s+1)+Fs(s-4)}{s(s-4)(s+1)}.$$

Setting $s=0$, $s=4$, and $s=-1$ gives $D=-3/4$, $E=3/20$, and $F=3/5$ (check). Finally, the third term is

$$\frac{s+3}{s(s-4)}=\frac{G}{s}+\frac{H}{s-4}=\frac{G(s-4)+Hs}{s(s-4)}.$$

Setting $s=0$ gives $G=-3/4$; setting $s=4$ gives $H=7/4$. Altogether,

$$Y(s)=\frac{1}{2}\frac{1}{s}+\frac{3}{2}\frac{1}{s-4}-2\frac{1}{s-3}+\frac{3}{4}\frac{1}{s}-\frac{3}{20}\frac{1}{s-4}-\frac{3}{5}\frac{1}{s+1}-\frac{3}{4}\frac{1}{s}+\frac{7}{4}\frac{1}{s-4}=\frac{1}{2}\frac{1}{s}+\frac{62}{20}\frac{1}{s-4}-\frac{13}{5}\frac{1}{s+1}.$$

Taking the inverse Laplace transform,

$$y(t)=\frac{1}{2}+\frac{31}{10}e^{4t}-\frac{13}{5}e^{-t}.$$

You should check that the answer actually satisfies the initial value problem.

(c) Taking the Laplace transform of both sides,

$$s^3Y(s)-s^2y(0)-sy'(0)-y''(0)+2s^2Y(s)-2sy(0)-2y'(0)-sY(s)-y(0)-2Y(s)=\frac{3}{s^2+9}.$$

Substituting the initial conditions and solving for $Y(s)$,

$$(s^3+2s^2-s-2)Y(s)=\frac{3}{s^2+9}+1.$$

We need to factor the polynomial on the left side at some point. Trying various divisors of -2 (the constant term) for s, we see that $s=1$ is a zero of the polynomial, so $s-1$ should be a factor. We can then easily factor the remaining quadratic to obtain

$$s^3+2s^2-s-2=(s-1)(s^2+3s+2)=(s-1)(s+1)(s+2).$$

Solving for $Y(s)$,

$$Y(s)=\frac{3}{(s-1)(s+1)(s+2)(s^2+9)}+\frac{1}{(s-1)(s+1)(s+2)}=\frac{s^2+12}{(s-1)(s+1)(s+2)(s^2+9)}.$$

(We may be able to save a little work by combining both terms into a single fraction and applying partial fractions just once.) Then

$$Y(s)=\frac{A(s+1)(s+2)(s^2+9)+B(s-1)(s+2)(s^2+9)+C(s-1)(s+1)(s^2+9)+(Ds+E)(s-1)(s+1)(s-}{(s-1)(s+1)(s+2)(s^2+9)}$$

Comparing numerators, and setting $s = 1$, $s = -1$, $s = -2$ and $s = 3i$, we have $A = 13/30$, $B = -13/20$, $C = -16/30$, $(3iD + E)(3i - 1)(3i + 1)(3i - 2) = 3$; the latter condition gives

$$(-9D - 6iD + 3iE - 2E)(-10) = 3 \implies 9D + 2E = \frac{3}{10} \text{ and } -6D + 3E = 0$$

which implies $D = 3/130$ and $E = 6/130$. Taking the inverse Laplace transform,

$$y(t) = \frac{13}{30}e^t - \frac{13}{20}e^{-t} - \frac{16}{30}e^{-2t} + \frac{3}{130}\cos 3t + \frac{2}{130}\sin 3t.$$

You should check that the above function satisfies the initial value problem.

4. (a) We have

$$\mathscr{L}\{(t-1)^2\}(s) = \mathscr{L}\{t^2 - 2t + 1\} = \frac{2}{s^3} - \frac{2}{s^2} + \frac{1}{s},$$

so by the formula for translation in the s-axis,

$$\mathscr{L}\{e^{2t}(t-1)^2\} = \mathscr{L}\{(t-1)^2\}(s-2) = \frac{2}{(s-2)^3} - \frac{2}{(s-2)^2} + \frac{1}{s-2}.$$

One way to check the result would be to apply the formula for translation in the t-axis to reduce to the Laplace transform of t^2e^{2t+2}, and then apply the formula for differentiation of a Laplace transform.

(b) We have

$$\mathscr{L}\{\cos 4t\}(s) = \frac{s}{s^2 + 16}$$

so by the formula for translation in the s-axis,

$$\mathscr{L}\{e^{-2t}\cos 4t\} = \mathscr{L}\{\cos 4t\}(s+2) = \frac{s+2}{(s+2)^2 + 16}.$$

I can't think of a straightforward way to check the result other than by using the definition of the Laplace transform (and integration by parts twice).

(c)

5. (a) The denominator of the argument can't be factored (over the reals) so we make do with completing the square instead:

$$\frac{1}{s^2 + 2s + 5} = \frac{1}{(s+1)^2 + 4}.$$

We have

$$\mathscr{L}\{\sin 2t\}(s) = \frac{2}{s^2 + 4}$$

so

$$\mathscr{L}\{e^{-t}\sin 2t\}(s) = \mathscr{L}\{\sin 2t\}(s+1) = \frac{2}{(s+1)^2 + 4}$$

and therefore

$$\mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2 + 4}\right\} = \frac{1}{2}e^{-t}\sin 2t.$$

You should check by taking the Laplace transform of the above.

(b) Again, we complete the square of the denominator:

$$\frac{2s + 5}{s^2 + 6s + 34} = \frac{2s + 5}{(s+3)^2 + 25}.$$

We want the numerator in the same form, as a function of $s+3$, so we write

$$\frac{2s+5}{(s+3)^2+25} = \frac{2(s+3)-1}{(s+3)^2+25} = 2\frac{s+3}{(s+3)^2+25} - \frac{1}{(s+3)^2+25}.$$

By the same technique as in the previous problem, we have

$$\mathscr{L}\left\{2\cos 5t - \frac{1}{5}\sin 5t\right\} = 2\frac{s}{s^2+25} - \frac{1}{s^2+25}$$

so

$$\mathscr{L}\left\{e^{-3t}\left(2\cos 5t - \frac{1}{5}\sin 5t\right)\right\}(s) = \mathscr{L}\left\{2\cos 5t - \frac{1}{5}\sin 5t\right\}(s+3) = 2\frac{(s+3)}{(s+3)^2+25} - \frac{1}{(s+3)^2+25}$$

and

$$\mathscr{L}^{-1}\left\{\frac{2s+5}{s^2+6s+34}\right\} = e^{-3t}\left(2\cos 5t - \frac{1}{5}\sin 5t\right).$$

Check by taking the Laplace transform of the above function.

(c) Here the denominator is already in terms of a power of $s-a$, so we only need to adjust the numerator:

$$\frac{(s+1)^2}{(s+2)^4} = \frac{((s+2)-1)^2}{(s+2)^4} = \frac{(s+2)^2-2(s+2)+1}{(s+2)^4} = \frac{1}{(s+2)^2} - \frac{2}{(s+2)^3} + \frac{1}{(s+2)^4}.$$

Then the inverse Laplace transform is

$$\mathscr{L}^{-1}\left\{\frac{(s+1)^2}{(s+2)^4}\right\} = te^{-2t} - t^2e^{-2t} + \frac{1}{6}t^3e^{-3t}.$$

You can check the answer by using differentiation of Laplace transforms.

6. (a) In order to apply the formula for translation in the t-axis, the argument of the function under the Laplace transform must match the argument of the step function. We write

$$\mathscr{L}\{e^{2-t}H(t-2)\} = \mathscr{L}\{e^{-(t-2)}H(t-2)\} = e^{-2s}\mathscr{L}\{e^{-t}\} = \frac{e^{-2s}}{s+1}.$$

The easiest way to check is by the definition of the Laplace transform.

(b) Again, the argument of the function under the Laplace transform must match the argument of the step function:

$$\mathscr{L}\{(3t+1)H(t-1)\} = \mathscr{L}\{(3(t-1)+4)H(t-1)\} = 3e^{-s}\mathscr{L}\{t\} + 4e^{-s}\mathscr{L}\{1\} = 3\frac{e^{-s}}{s^2} + 4\frac{e^{-s}}{s}.$$

Again, the definition of the Laplace transform is the easiest way to check the answer.

(c) We need to rewrite the argument of sin in terms of $t-\pi/2$ by applying the angle addition identity:

$$\sin(t) = \sin\left(\left(t-\frac{\pi}{2}\right)+\frac{\pi}{2}\right) = \sin\left(t-\frac{\pi}{2}\right)\cos\frac{\pi}{2} + \cos\left(t-\frac{\pi}{2}\right)\sin\frac{\pi}{2} = \cos\left(t-\frac{\pi}{2}\right).$$

Therefore

$$\mathscr{L}\left\{(\sin t)H\left(t-\frac{\pi}{2}\right)\right\} = \mathscr{L}\left\{\cos\left(t-\frac{\pi}{2}\right)H\left(t-\frac{\pi}{2}\right)\right\} = e^{-\pi s/2}\mathscr{L}\{\cos t\} = \frac{se^{-\pi s/2}}{s^2+1}.$$

You should check your answer.

7. (a) Expanding the numerator of the function under the Laplace transform,

$$\frac{(1+e^{-2s})^2}{s+2} = \frac{1}{s+2} + 2\frac{e^{-2s}}{s+2} + \frac{e^{-4s}}{s+2}.$$

Then by the linearity of the inverse Laplace transform and the formula for translation in the t-axis,

$$\mathscr{L}^{-1}\left\{\frac{(1+e^{-2s})^2}{s+2}\right\} = e^{-2t} + 2e^{-2(t-2)}H(t-2) + e^{-2(t-4)}H(t-4).$$

You should check the answer by taking the Laplace transform of the above.

(b) This is similar to 6(c) above.

(c) Apply partial fractions to

$$\frac{1}{s^2(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} = \frac{As(s-1)+B(s-1)+Cs^2}{s^2(s-1)}.$$

Setting $s=0$ gives $B=-1$, and setting $s=1$ gives $C=1$. Note that the substitution trick doesn't give us A. We can get A by taking derivatives of the numerators and applying the substitution trick, but it's easier in this case just to compare coefficients of s^2 on both sides to get $A+C=0$, $A=-1$. Now we can write

$$\frac{e^{-2s}}{s^2(s-1)} = -\frac{e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{e^{-2s}}{s-1}.$$

Using the formula for translation in the t-axis and taking the inverse Laplace transform,

$$\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s^2(s-1)}\right\} = -(t-2)^0H(t-2) - (t-2)^1H(t-2) + e^{t-2}H(t-2) = (1-t+e^{t-2})H(t-2).$$

You should check the answer by taking the Laplace transform of the above.