MATH281 200610 Problem Set 11 Solutions DRAFT

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Wednesday, April 12, 2006

1. While you could use the definition of the Laplace transform, it is easier to represent each of the given functions in terms of the Heaviside step function and then use the formula for translation in the t-axis.

   (a) Here we have

   $$f(t) = 1 - H(t-4) + H(t-5) = (t-0)^0 - (t-4)^0 H(t-4) + (t-5)^0 H(t-5).$$

   By the linearity of the Laplace transform and translation in the t-axis,

   $$\mathscr{L}\{f(t)\} = \mathscr{L}\{(t-0)^0\} - \mathscr{L}\{(t-4)^0 H(t-4)\} + \mathscr{L}\{(t-5)^0 H(t-5)\} = \frac{1}{s} - e^{-4s}\frac{1}{s} + e^{-5s}\frac{1}{s}.$$

   (b) Here we need to cancel out the sin function from 2π onwards, so

   $$f(t) = \sin t - (\sin t)H(t-2\pi) = \sin t - \sin(t-2\pi)H(t-2\pi)$$

   by the periodicity of the $\sin t$. By the linearity of the Laplace transform and translation in the t-axis,

   $$\mathscr{L}\{f(t)\} = \mathscr{L}\{\sin t\} - \mathscr{L}\{\sin(t-2\pi)H(t-2\pi)\} = \frac{1}{s^2+1} - e^{-2\pi s}\frac{1}{s^2+1}.$$

   (c) We can represent $f(t)$ as an infinite sum of step functions:

   $$f(t) = H(t-1) + H(t-2) + H(t-3) + \cdots.$$

   Now assuming the technicality of convergence in some appropriate sense, we can apply linearity and translation in the t-axis to obtain

   $$\mathscr{L}\{f(t)\} = \frac{e^{-1s}}{s} + \frac{e^{-2s}}{s} + \frac{e^{-3s}}{s} + \cdots = \frac{1}{s}(e^{-1s} + e^{-2s} + e^{-3s} + \cdots).$$

   The answer can be simplified by noting that the infinite series is a geometric series ($x = e^{-s}$):

   $$x(1 + x + x^2 + \cdots) = \frac{x}{1-x}$$

   by the rule for summation of infinite geometric series. Therefore we can write

   $$\mathscr{L}\{f(t)\} = \frac{e^{-s}}{(1-e^{-s})s}.$$

2. (a) First, we write $f(t)$ in terms of the Heaviside step function:

   $$f(t) = 1 - 2H(t-1).$$

   The Laplace transform of $f(t)$ is

   $$\mathscr{L}\{f(t)\} = (1 - 2e^{-s})\frac{1}{s}.$$

Taking the Laplace transform of both sides of the differential equation,

$$sY(s) - y(0) + Y(s) = (1 - 2e^{-s})\frac{1}{s}.$$

Using the initial condition $y(0) = 0$, solving for $Y(s)$, and finding the partial fractions decomposition,

$$Y(s) = (1 - e^{-s})\frac{1}{s(s+1)} = (1 - 2e^{-s})\left(\frac{1}{s} - \frac{1}{s+1}\right) = \frac{1}{s} - \frac{1}{s+1} - 2e^{-s}\frac{1}{s} + 2e^{-s}\frac{1}{s+1}.$$

Taking the inverse Laplace transform,

$$y(t) = 1 - e^{-t} - 2(1 - e^{-(t-1)})H(t-1).$$

You should check that the above function satisfies the differential equation on the intervals $(0, 1)$ and $(1, \infty)$, is continuous at $t = 1$, and satisfies the initial condition $y(0) = 0$. Alternatively, for the sake of comparison, you could try solving the problem by solving it separately on $(0, 1)$ and $(1, \infty)$, and finding particular solutions that satisfy the initial condition and match at $t = 1$.

(b) The function f can be written

$$f(t) = 1 - H(t-1)$$

with Laplace transform

$$F(s) = (1 - e^{-s})\frac{1}{s}$$

Taking the Laplace transform of the differential equation,

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = (1 - e^{-s})\frac{1}{s}.$$

Using the initial condition, solving for $Y(s)$, and applying partial fractions,

$$Y(s) = -\frac{1}{s^2+4} + (1 - e^{-s})\frac{1}{s(s^2+4)} = -\frac{1}{s^2+4} + (1 - e^{-s})\frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right).$$

Taking the inverse Laplace transform,

$$y(t) = -\frac{1}{2}\sin 2t + \frac{1}{4}(1 - \cos 2t) - \frac{1}{4}(1 - \cos 2(t-1))H(t-1).$$

You should check that the above function satisfies the differential equation on $(0, 1)$ and $(1, \infty)$, satisfies the initial conditions at $t = 0$, and that the function and its first derivative are continuous at $t = 1$. For the sake of comparison, try solving the problem using undetermined coefficients.

(c) Taking the Laplace transform of the differential equation,

$$s^2Y(s) - sy(0) - y'(0) - 5sY(s) - 5y(0) + 6Y(s) = \frac{e^{-s}}{s}.$$

Using the initial conditions and solving for $Y(s)$,

$$Y(s) = \frac{1}{s^2 - 5s + 6} + \frac{e^{-s}}{s(s^2 - 5s + 6)}.$$

In order to expand the fractions above as partial fractions, we need to factor the denominators. We have

$$s^2 - 5s + 6 = (s-2)(s-3)$$

so

$$\frac{1}{s^2 - 5s + 6} = \frac{1}{s-3} - \frac{1}{s-2}$$

and (saving a bit of work by making use of the above partial fractions decomposition)

$$\frac{1}{s(s^2-5s+6)} = \frac{1}{s(s-3)} - \frac{1}{s(s-2)} = \frac{1}{3}\frac{1}{s-3} - \frac{1}{3}\frac{1}{s} + \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s-2}.$$

In summary,

$$Y(s) = \frac{1}{s-3} - \frac{1}{s-2} + e^{-s}\left(\frac{1}{3}\frac{1}{s-3} + \frac{1}{6}\frac{1}{s} - \frac{1}{2}\frac{1}{s-2}\right).$$

Taking the inverse Laplace transform,

$$y(t) = e^{3t} - e^{2t} + \left(\frac{1}{3}e^{3(t-1)} - \frac{1}{2}e^{2(t-1)} + \frac{1}{6}\right) H(t-1).$$

You should check that the above function really is a solution to the initial value problem. It should satisfy the differential equation at the points where the input function is continuous, and should be C^1 at points where the input function is discontinuous, and it should satisfy the initial conditions.

3. (a) Using the formula for the transform of a derivative,

$$\mathscr{L}\{t^3e^t\}(s) = (-1)^3\frac{d^3}{ds^3}\mathscr{L}\{e^t\}(s) = -\frac{d^3}{ds^3}(s-1)^{-1} = 6(s-1)^{-4}.$$

You could check by using the definition of the Laplace transform and integration by parts, but it would be rather involved. You could speed up the work considerably by using a reduction formula for the integral in questions, which you could find in any good table of integrals.

(b) By the formula for the transform of a derivative,

$$\mathscr{L}\{t^2\cos t\}(s) = (-1)^2\frac{d^2}{ds^2}\mathscr{L}\{\cos t\}(s) = \frac{d}{ds}\frac{d}{ds}\frac{s}{s^2+1} = \frac{d}{ds}\frac{1-s^2}{(s^2+1)^2} = \frac{2s^3-6s}{(1+s^2)^3}.$$

Again, you could check using a reduction formula for the integral of $t^n \cos t$.

(c) By the formula for the derivative of a Laplace transform,

$$\mathscr{L}\{te^{-3t}\cos 3t\}(s) = -\frac{d}{ds}\mathscr{L}\{e^{-3t}\cos 3t\}(s).$$

You may be able to look up the latter Laplace transform in a table, or you could apply the formula for translation in the s-axis:

$$\mathscr{L}\{e^{-3t}\cos 3t\}(s) = \mathscr{L}\{\cos 3t\}(s+3) = \frac{s+3}{(s+3)^2+9}.$$

Substituting into the first identity,

$$\mathscr{L}\{te^{-3t}\cos 3t\}(s) = -\frac{d}{ds}\frac{s+3}{(s+3)^2+9} = \frac{(s+3)^2-9}{((s+3)^2+9)^2}.$$

You may be able to check the above result with the aid of an even better table of integrals.

4. (a) By the formula for the Laplace transform of a convolution,

$$\mathscr{L}\{t^2 * te^t\}(s) = \mathscr{L}\{t^2\}(s)\cdot\mathscr{L}\{te^t\}(s) = \frac{2}{s^3}\cdot -\frac{d}{ds}\mathscr{L}\{e^t\}(s) = \frac{2}{s^3}\cdot\frac{1}{(s-1)^2}.$$

You may be able to check the above result by evaluating the convolution explicitly and calculating the Laplace transform by other methods.

(b) Just by looking at the problem, the answer should be

$$\mathscr{L}\{e^{2t} * \sin t\} = \frac{1}{s-2} \cdot \frac{1}{s^2+1}.$$

You may be able to check by evaluating the convolution explicitly.

(c) You should be able to identify this as

$$\mathscr{L}\{\sin t * \cos t\} = \frac{s}{(s^2+1)^2}.$$

You may be able to check by evalutating the integral explicitly.

5. The inverse form of the Laplace transform of an integral may be written

$$\mathscr{L}^{-1}\left\{\frac{F(s)}{s}\right\}(t) = \int_0^t \mathscr{L}^{-1}\{F(s)\}(\tau)\, d\tau.$$

(a) Applying the above formula,

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s-1)}\right\}(t) = \int_0^t \mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\}(\tau)\, d\tau = \int_0^t e^{\tau}\, d\tau = e^t - 1.$$

You can check your answer using partial fractions.

(b) Applying the formula for the inverse form of the Laplace transform of an integral and using the previous result,

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2(s-1)}\right\}(t) = \int_0^t \mathscr{L}^{-1}\left\{\frac{1}{s(s-1)}\right\}(\tau)\, d\tau = \int_0^t e^{\tau} - 1\, d\tau = e^t - t - 1.$$

You can check the answer using partial fractions.

(c) The straightforward way of answering this question requires finding

$$\mathscr{L}^{-1}\left\{\frac{1}{(s-a)^2}\right\}(t) = \mathscr{L}^{-1}\left\{-\frac{d}{ds}\frac{1}{s-a}\right\}(t) = te^{at}$$

by the formula for the derivative of a Laplace transform. Now by the inverse form of the Laplace transform of an integral,

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s-a)^2}\right\}(t) = \int_0^t \mathscr{L}^{-1}\left\{\frac{1}{(s-a)^2}\right\}(\tau)\, d\tau = \int_0^t \tau e^{a\tau}\, d\tau.$$

Integrating by parts,

$$\int_0^t \tau e^{a\tau}\, d\tau = \tau\frac{1}{a}e^{a\tau}\bigg|_0^t - \int_0^t \frac{1}{a}e^{a\tau}\, d\tau = \tau\frac{1}{a}e^{a\tau} - \frac{1}{a^2}e^{a\tau}\bigg|_0^t = \frac{1}{a}te^{at} - \frac{1}{a^2}e^{at} + \frac{1}{a^2}$$

which is the desired inverse Laplace transform. You could check by partial fractions. Alternatively, you could have applied translation in the s-axis to reduce the problem to

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s-a)^2}\right\}(t) = e^{at}\mathscr{L}^{-1}\left\{\frac{1}{(s+a)(s+a-a)^2}\right\}(t) = e^{at}\mathscr{L}^{-1}\left\{\frac{1}{s^2(s+a)}\right\}(t),$$

i.e., something very much like the previous problem 5(b).