MATH281 200610 Problem Set 12 Solutions DRAFT

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1. Write the system of equations in question as $X'(t) = AX(t) + F(t)$, and write

$$X_1 = \begin{bmatrix} e^{\sqrt{2}t} \\ (-1-\sqrt{2})e^{\sqrt{2}t} \end{bmatrix},$$
$$X_2 = \begin{bmatrix} e^{-\sqrt{2}t} \\ (-1+\sqrt{2})e^{-\sqrt{2}t} \end{bmatrix},$$
$$X_p = \begin{bmatrix} t^2 - 2t + 1 \\ 4t \end{bmatrix}.$$

We need to show that X_1 and X_2 are linearly independent solutions to the homogeneous equation $X' = AX$, and that X_p is a particular solution. The result then follows from the theory of first order systems. First, X_1 is a solution to the homogenous system because

$$X_1' = \begin{bmatrix} \sqrt{2}e^{\sqrt{2}t} \\ (-\sqrt{2}-2)e^{\sqrt{2}t} \end{bmatrix}$$

and

$$\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} X = \begin{bmatrix} (-1)e^{\sqrt{2}t} + (-1)(-1-\sqrt{2})e^{\sqrt{2}t} \\ (-1)e^{\sqrt{2}t} + (1)(-1-\sqrt{2})e^{\sqrt{2}t} \end{bmatrix} = \begin{bmatrix} \sqrt{2}e^{\sqrt{2}t} \\ (-\sqrt{2}-2)e^{\sqrt{2}t} \end{bmatrix}$$

are equal. Similarly, it is straightforward to check that X_2 is a solution to the homogeneous system. The solutions X_1 and X_2 are linearly independent because the Wronskian

$$W = \begin{vmatrix} X_1 & X_2 \end{vmatrix} = \begin{vmatrix} e^{\sqrt{2}t} & e^{-\sqrt{2}t} \\ (-1-\sqrt{2})e^{\sqrt{2}t} & (-1+\sqrt{2})e^{-\sqrt{2}t} \end{vmatrix} = (-1+\sqrt{2}) - (-1-\sqrt{2}) = 2\sqrt{2}$$

is never zero. Finally, X_p is a particular solution because the left hand side of the equation is

$$X_p' = \begin{bmatrix} 2t - 2 \\ 4 \end{bmatrix}$$

while the right hand side is

$$\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} t^2 - 2t + 1 \\ 4t \end{bmatrix} + \begin{bmatrix} t^2 + 4t - 1 \\ t^2 - 6t + 5 \end{bmatrix} = \begin{bmatrix} -t^2 - 2t - 1 \\ -t^2 + 6t - 1 \end{bmatrix} + \begin{bmatrix} t^2 + 4t - 1 \\ t^2 - 6t + 5 \end{bmatrix} = \begin{bmatrix} 2t - 2 \\ 4 \end{bmatrix}$$

and the two sides agree. The desired conclusion follows.

2. (a) We write the system in matrix form:

$$X' = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} X.$$

To solve the system we find two eigenvalues (and their corresponding eigenvectors) of the coefficient matrix. The characteristic polynomial is

$$\begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 4 = (\lambda-1)(\lambda-4)$$

so the eigenvalues are $\lambda = 1, 4$. An eigenvector corresponding to $\lambda = 1$ may be found by solving the system

$$\begin{bmatrix} 2-1 & 2 \\ 1 & 3-1 \end{bmatrix} V_1 = O$$

where O is the zero vector. One solution is

$$V_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

and the corresponding solution to the differential equation is

$$X_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^t.$$

You should check that X_1 really is a solution to the differential equation. Similarly, an eigenvector corresponding to the eigenvalue $\lambda = 4$ can be found by solving

$$\begin{bmatrix} 2-4 & 2 \\ 1 & 3-4 \end{bmatrix} V_2 = O$$

which gives the solution

$$X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{4t}.$$

You should check that X_2 really is a solution, and that X_1 and X_2 are linearly independent. It follows that the general solution to the differential equation is $X = c_1X_1 + c_2X_2$.

(b) The characteristic equation for the coefficient matrix is $(-6-\lambda)(1-\lambda)+6 = \lambda^2 + 5\lambda = 0$ with solutions $\lambda = 0, -5$. The corresponding eigenvectors are

$$V_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, V_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

so the general solution to the system is

$$X = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{0t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-5t} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-5t}.$$

You should check that the above really is a solution, and that setting $c_1 = 1$, $c_2 = 0$, and setting $c_1 = 0$, $c_2 = 1$ give two linearly independent solutions.

(c) The coefficient matrix is

$$\begin{bmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{bmatrix}.$$

The characteristic equation is

$$\begin{vmatrix} 2-\lambda & -7 & 0 \\ 5 & 10-\lambda & 4 \\ 0 & 5 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 10-\lambda & 4 \\ 5 & 2-\lambda \end{vmatrix} + 7\begin{vmatrix} 5 & 4 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2-12\lambda)+35(2-\lambda).$$

The characteristic polynomial factors as

$$(2-\lambda)(\lambda^2-12\lambda)+35(2-\lambda). = (2-\lambda)(\lambda^2-12\lambda+35) = (2-\lambda)(\lambda-5)(\lambda-7),$$

so the eigenvalues are $\lambda = 2, 5, 7$. An eigenvector corresponding to $\lambda = 2$ is a solution to the system with augmented matrix

$$\left[\begin{array}{ccc|c} 0 & -7 & 0 & 0 \\ 5 & 8 & 4 & 0 \\ 0 & 5 & 0 & 0 \end{array}\right].$$

We can solve the system in a number of ways; I prefer row reduction. Multiplying row 1 by $-1/7$ and row 3 by $1/5$ gives

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 5 & 8 & 4 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right].$$

Adding -8 times row 1 to row 2 and -1 times row 1 to row 3 gives

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 5 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Now we can see that a solution to the system is

$$V_1 = \left[\begin{array}{c} 4 \\ 0 \\ -5 \end{array}\right].$$

You should check that V_1 really is an eigenvector of the original coefficient matrix. Similarly, to find an eigenvector corresponding to $\lambda = 5$, we solve the system with augmented matrix

$$\left[\begin{array}{ccc|c} -3 & -7 & 0 & 0 \\ 5 & 5 & 4 & 0 \\ 0 & 5 & -3 & 0 \end{array}\right].$$

I prefer to deal with integers only when doing row reduction, which should explain the following odd manoeuvers. Adding 1 times row 1 to row 2 gives

$$\left[\begin{array}{ccc|c} -3 & -7 & 0 & 0 \\ 2 & -2 & 4 & 0 \\ 0 & 5 & -3 & 0 \end{array}\right].$$

Multiplying row 2 by $1/2$ and swapping rows 1 and 2 gives

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ -3 & -7 & 0 & 0 \\ 0 & 5 & -3 & 0 \end{array}\right].$$

Adding 3 times row 1 to row 2 gives

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & -10 & 6 & 0 \\ 0 & 5 & -3 & 0 \end{array}\right].$$

Adding 2 times row 3 to row 2 and then swapping rows 2 and 3 gives

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 5 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Multiplying row 1 by 5 and then adding 1 times row 2 to row 1 gives

$$\left[\begin{array}{ccc|c} 5 & 0 & 7 & 0 \\ 0 & 5 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore an eigenvector corresponding to eigenvalue 5 is

$$V_2 = \begin{bmatrix} -7 \\ 3 \\ 5 \end{bmatrix}$$

(check). Finally, an eigenvector corresponding to the eigenvalue $\lambda = 7$ is a solution to the system with augmented matrix

$$\left[\begin{array}{ccc|c} -5 & -7 & 0 & 0 \\ 5 & 3 & 4 & 0 \\ 0 & 5 & -5 & 0 \end{array}\right].$$

Adding 1 times row 1 to row 2 gives

$$\left[\begin{array}{ccc|c} -5 & -7 & 0 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 5 & -5 & 0 \end{array}\right].$$

Multiplying row 2 by $-1/4$ and adding -7 times the resulting row to row 1 and -5 times the resulting row to row 3 gives

$$\left[\begin{array}{ccc|c} -5 & 0 & -7 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

You can see from the above row echelon form that an eigenvector is

$$V_3 = \begin{bmatrix} 7 \\ 5 \\ 5 \end{bmatrix}$$

(check). Altogether, the general solution to the system is

$$X = c_1 \begin{bmatrix} 4 \\ 0 \\ -5 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -7 \\ 3 \\ 5 \end{bmatrix} e^{5t} + c_3 \begin{bmatrix} 7 \\ 5 \\ 5 \end{bmatrix} e^{7t}.$$

You might want to check that the above really is a solution (although that check isn't necessary if you've checked all the eigenvectors). You should also think of why you can get three linearly independent solutions from the above, but if you know the linear algebra of eigenvectors, you'll see a short cut so that you don't have to evaluate the Wronskian.

3. (a) The characteristic equation is

$$(1-\lambda)^3 - (1-\lambda) = (1-\lambda)\lambda(\lambda-2) = 0$$

so the eigenvalues are $\lambda = 0, 1, 2$, with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

The general solution to the system is therefore

$$X(t) = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}.$$

(Why can we get three linearly independent solutions from the above?) Rewriting the above in matrix notation,

$$X(t) = \begin{bmatrix} 1 & 0 & e^{2t} \\ 0 & e^t & 0 \\ -1 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

The square matrix above is a fundamental matrix $\Phi(t)$; any particular solution can be written as $\Phi(t)C$ where C is a column vector of constants. To find a particular solution satisfying the initial value condition, note that

$$X(0) = \Phi(0)C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}.$$

Solving for C, the system has augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & -4 \end{array}\right].$$

Adding 1 times row 1 to row 3,

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -2 \end{array}\right].$$

Multiplying row 3 by 1/2 and then adding -1 times row 3 to row 1 gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

So we have $c_1 = 3$, $c_2 = 0$, $c_3 = -1$ (check), and the solution to the initial value problem can be written

$$X(0) = \Phi(t)C = \begin{bmatrix} 1 & 0 & e^{2t} \\ 0 & e^t & 0 \\ -1 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}.$$

(b) This problem is similar to the previous.

4. (a) After forming A^2, A^3, etc., you should see the pattern

$$A^n = \begin{bmatrix} 3^n & 0 \\ 0 & (-2)^n \end{bmatrix}.$$

Therefore

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \cdots = \begin{bmatrix} 1 + 3t + (1/2!)(3t)^2 + \cdots & 0 \\ 0 & 1 - 2t + (1/2!)(2t)^2 + \cdots \end{bmatrix} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-2t} \end{bmatrix}.$$

You can check by differentiating: you should have $(e^{At})' = Ae^{At}$. The reciprocal exponential e^{-At} can be determined from the above by replacing t with $-t$. A similar calculation applies for the exponential of any diagonal matrix.

(b) The first few powers of A are

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and so on. In general, $A^k = I$ if k is even and A if k is odd. Therefore the matrix exponential is

$$e^{At} = \begin{bmatrix} 1 + (1/2!)t^2 + (1/4!)t^4 + \cdots & t + (1/3!)t^3 + (1/5!)t^5 + \cdots \\ t + (1/3!)t^3 + (1/5!)t^5 + \cdots & 1 + (1/2!)t^2 + (1/4!)t^4 + \cdots \end{bmatrix}.$$

In order to find closed form expressions for the above series, write

$$f_1(t) = 1 + (1/2!)t^2 + (1/4!)t^4 + \cdots$$
$$f_2(t) = t + (1/3!)t^3 + (1/5!)t^5 + \cdots.$$

Then $f_1(t) + f_2(t) = e^t$ and $f_1(t) - f_2(t) = e^{-t}$ so

$$f_1(t) = \frac{e^t + e^{-t}}{2} = \cosh t, \quad f_2(t) = \frac{e^t - e^{-t}}{2} = \sinh t.$$

So we can write

$$e^{At} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}, \quad e^{-At} = \begin{bmatrix} \cosh(-t) & \sinh(-t) \\ \sinh(-t) & \cosh(-t) \end{bmatrix} = \begin{bmatrix} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{bmatrix}.$$

You can check by differentiating; also, we should have $e^{At}e^{-At} = I$.

(c) Once again, we try to find a pattern in the powers of the given matrix A. We have

$$A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and so on for all higher powers of A. In this case (nilpotent A), the exponential is a finite series, and we have

$$e^{At} = \begin{bmatrix} 1 & 0 & 0 \\ 3t & 1 & 0 \\ 5t + 3t^2 & t & 1 \end{bmatrix}, \quad e^{-At} = \begin{bmatrix} 1 & 0 & 0 \\ -3t & 1 & 0 \\ -5t + 3t^2 & -t & 1 \end{bmatrix},$$

You can check by differentiating and by multiplying the above two matrices.

The above method only works if we can find powers A^k of a matrix A. In the generic case, powers can be found by diagonalization (essentially, by finding all the eigenvalues and eigenvectors). In the case of repeated eigenvalues, however, like problem (c) above, we would have to use Jordan canonical form, which is fussy and numerically unstable. The method demonstrated below is an attempt to mitigate the difficulties that arise when performing diagonalization.

5. (a) You could try to find the exponential by diagonalization, but it is probably better to use the inverse Laplace transform method. We find in succession

$$sI - A = \begin{bmatrix} s-4 & 2 \\ -1 & s-1 \end{bmatrix},$$

the inverse matrix

$$(sI - A)^{-1} = \frac{1}{(s-4)(s-1)+2} \begin{bmatrix} s-1 & -2 \\ 1 & s-4 \end{bmatrix} = \begin{bmatrix} \frac{s-1}{(s-2)(s-3)} & -\frac{2}{(s-2)(s-3)} \\ \frac{1}{(s-2)(s-3)} & \frac{s-4}{(s-2)(s-3)} \end{bmatrix},$$

and the inverse Laplace transform of the inverse matrix

$$\mathscr{L}^{-1}\{(sI - A)^{-1}\} = \mathscr{L}^{-1}\left\{\begin{bmatrix} 2\frac{1}{s-3} - \frac{1}{s-2} & -2\frac{1}{s-3} + 2\frac{1}{s-2} \\ \frac{1}{s-3} + \frac{1}{s-2} & -\frac{1}{s-3} + 2\frac{1}{s-2} \end{bmatrix}\right\} = \begin{bmatrix} 2e^{3t} - e^{2t} & -2e^{3t} + 2e^{2t} \\ e^{3t} + e^{2t} & -e^{3t} + 2e^{2t} \end{bmatrix}.$$

You should check that the above matrix $\Phi(t)$ is a fundamental solution to the differential equation, i.e., that $\Phi'(t) = A\Phi(t)$ and that $\Phi(t)$ is invertible for all t. Then the general solution to the differential equation is $X(t) = \Phi(t)C$ where C is a column vector of constants, and a solution to the initial value problem is $X(t) = \Phi(t)X_0$. (Note that when we use the matrix exponential as the fundamental matrix, we don't have to solve a system to find a particular solution to an initial value problem. That is because in this case $\Phi(0) = e^{A\cdot 0} = e^O = I$.)

(b) In this case, diagonalizing the matrix would require dealing with complex eigenvalues, which we didn't cover. But we can still solve the problem using the Laplace transform method. In succession,

$$sI - A = \begin{bmatrix} s & -1 \\ 2 & s+2 \end{bmatrix},$$

and the inverse is

$$(sI - A)^{-1} = \frac{1}{s(s+2)+2} \begin{bmatrix} s+2 & 1 \\ -2 & s \end{bmatrix} = \begin{bmatrix} \frac{s+2}{(s+1)^2+1} & \frac{1}{(s+1)^2+1} \\ -\frac{2}{(s+1)^2+1} & \frac{s}{(s+1)^2+1} \end{bmatrix},$$

and the inverse Laplace transform of the inverse matrix

$$\mathscr{L}^{-1}\{(sI - A)^{-1}\} = \mathscr{L}^{-1}\left\{\begin{bmatrix} \frac{(s+1)+1}{(s+1)^2+1} & \frac{1}{(s+1)^2+1} \\ -\frac{2}{(s+1)^2+1} & \frac{(s+1)-1}{(s+1)^2+1} \end{bmatrix}\right\} = \begin{bmatrix} e^{-t}(\cos t + \sin t) & e^{-t}\sin t \\ -2e^{-t}\sin t & e^{-t}(\cos t - \sin t) \end{bmatrix}.$$

You should check that the above is a fundamental matrix for the system with the property that $\Phi(0) = I$. Then the general solution is $X(t) = \Phi(t)C$, and the solution to the IVP is $X(t) = \Phi(t)X_0$.

(c) Again, diagonalizing the matrix or solving the system $X' = AX$ by eigenvalues would require material we did not cover, namely repeated eigenvalues. So we must resort to the Laplace transform method. First we form

$$sI - A = \begin{bmatrix} s & 0 & -1 \\ 0 & s-1 & 0 \\ -1 & 0 & s \end{bmatrix}.$$

To invert the above matrix we could use row reduction, but I prefer Cramer's rule in cases like this. The determinant of the matrix is

$$\det(sI - A) = s^2(s-1) - (s-1) = (s-1)(s^2-1) = (s-1)^2(s+1).$$

The matrix of minors M_{ij} obtained by taking the determinant of the 2×2 matrix left when row i and column j is deleted is

$$M = \begin{bmatrix} s(s-1) & 0 & s-1 \\ 0 & s^2-1 & 0 \\ s-1 & 0 & s(s-1) \end{bmatrix}.$$

The matrix of cofactors $C_{ij} = (-1)^{i+j}M_{ij}$ obtained from the matrix of minors by negating entries in positions where $i+j$ is odd is

$$C = \begin{bmatrix} s(s-1) & 0 & s-1 \\ 0 & s^2-1 & 0 \\ s-1 & 0 & s(s-1) \end{bmatrix},$$

which coincidentally is the same as the matrix of minors. The classical adjoint of $(sI - A)$ is the transpose of the matrix of cofactors,

$$\operatorname{adj}(sI - A) = \begin{bmatrix} s(s-1) & 0 & s-1 \\ 0 & s^2-1 & 0 \\ s-1 & 0 & s(s-1) \end{bmatrix},$$

which again is coincidentally the same as the matrix of minors. Finally, the inverse is the classical adjoint divided by the determinant, i.e.

$$(sI - A)^{-1} = \begin{bmatrix} \frac{s}{(s-1)(s+1)} & 0 & \frac{1}{(s-1)(s+1)} \\ 0 & \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)(s+1)} & 0 & \frac{s}{(s-1)(s+1)} \end{bmatrix}.$$

Taking the inverse Laplace transform of the above,

$$e^{At} = \mathscr{L}^{-1}\left\{\begin{bmatrix} \frac{1/2}{s-1} + \frac{1/2}{s+1} & 0 & \frac{1/2}{s-1} - \frac{1/2}{s+1} \\ 0 & \frac{1}{s-1} & 0 \\ \frac{1/2}{s-1} - \frac{1/2}{s+1} & 0 & \frac{1/2}{s-1} + \frac{1/2}{s+1} \end{bmatrix}\right\} = \begin{bmatrix} \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} & 0 & \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} \\ 0 & e^{t} & 0 \\ \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} & 0 & \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} \end{bmatrix}.$$

We can also write

$$e^{At} = \begin{bmatrix} \cosh t & 0 & \sinh t \\ 0 & e^{t} & 0 \\ \sinh t & 0 & \cosh t \end{bmatrix}.$$

Check. Isn't that neat?